THE EIGENVALUE PROBLEM FOR A CLASS OF LINEAR INTEGRAL OPERATORS. MAS214: LODE

Let \( L = \{ f(x), \ a \leq x \leq b \} \) be the space of continuous functions on \([a, b]\). A linear integral operator \( A \) with kernel \( K(x, y) \) is defined by

\[
(Af)(x) = \int_a^b K(x, y)f(y)dy,
\]

where \( K(x, y) \) is a continuous function of \((x, y), \ a \leq x, y \leq b\).

**Definition 1.** \( f \) is an eigenfunction of \( A \) if \( Af = \lambda f \), and \( \lambda \) is the eigenvalue corresponding to the eigenfunction \( f \).

In our case the eigenvalue-eigenfunction equation reads

\[
(Af)(x) = \int_a^b K(x, y)f(y)dy = \lambda f(x),
\]

where \( f(x) \) and \( \lambda \) are the unknowns. However, the task of solving this equation with arbitrary \( K(x, y) \) is far too difficult if not impossible. In these notes we consider a very particular but useful case when \( K(x, y) \) is a finite sum of products of continuous functions \( \phi_j(x)g_j(y) \):

\[
K(x, y) = \sum_{j=1}^{n} \phi_j(x)g_j(y).
\]  

(1)

We shall make use of the following notations:

\[ a_{jk} = \int_a^b g_j(y)\phi_k(y)dy \]

**Theorem 1.** Suppose that functions \( \phi_j(x), \ 1 \leq j \leq n \) are linearly independent. Then

(i) \( f(x) \) is the eigenfunction with zero eigenvalue if and only if

\[
\int_a^b g_j(y)f(y)dy = 0 \text{ for all } j, \ 1 \leq j \leq n.
\]  

(2)

(ii) \( \lambda \neq 0 \) is a non-zero eigenvalue of \( A \) if and only if it satisfies the following equation:

\[
\det \begin{pmatrix}
    a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda
\end{pmatrix} = 0
\]  

(3)

and the relevant eigenfunction is of the form

\[
f(x) = \sum_{j=1}^{n} c_j \phi_j(x),
\]
where \(c_1, \ldots, c_n\) is a (non-trivial) solution of the following system of linear equations:

\[
\begin{align*}
(a_{11} - \lambda)c_1 + a_{12}c_2 + \ldots + a_{1n}c_n &= 0 \\
a_{21}c_1 + (a_{22} - \lambda)c_2 + \ldots + a_{2n}c_n &= 0 \\
&\quad \vdots \\
a_{n1}c_1 + a_{n2}c_2 + \ldots + (a_{nn} - \lambda)c_n &= 0
\end{align*}
\]

(4)

Remarks. 1. The easy consequence of this theorem is that zero is always among the finite number of the eigenvalues of our operator. Indeed, relations (2) tells us that any function \(f(x)\) which is orthogonal to all functions \(g_j(x)\) is an eigenfunction of this operator with eigenvalue 0. The existence of (infinitely many) such functions follows for instance from the orthogonalization procedure.

2. Since (3) is a polynomial equation and the polynomial in the lhs of (3) is of degree \(n\), the operator \(A\) has no more than \(n\) non-zero eigenvalues in total.

3. If the only solution to (3) is zero, then 0 is the only eigenvalue of \(A\).

Proof of Theorem 1.

(i) If \(f(x)\) is an eigenfunction with zero eigenvalue then by definition

\[(Af)(x) = \int_a^b K(x, y)f(y)dy = 0\]

or

\[\int_a^b K(x, y)f(y)dy = \sum_{j=1}^n \phi_j(x) \int_a^b g_j(y)f(y)dy = 0 \text{ for all } x.\]

Since functions \(\phi_j(x)\) are linearly independent the last equation holds if and only if all relations (2) are satisfied. QED

(ii) If \(\lambda \neq 0\) is an eigenvalue of \(A\), then

\[(Af)(x) = \int_a^b K(x, y)f(y)dy = \lambda f(x)\]

or

\[\int_a^b K(x, y)f(y)dy = \sum_{j=1}^n \phi_j(x) \int_a^b g_j(y)f(y)dy = \lambda f(x).\]

Hence

\[f(x) = \lambda^{-1} \sum_{j=1}^n \phi_j(x) \int_a^b g_j(y)f(y)dy = \sum_{j=1}^n c_j \phi_j(x),\]

(6)

where \(c_j = \lambda^{-1} \int_a^b g_j(y)f(y)dy\) which can also be re-written as

\[\int_a^b g_j(y)f(y)dy = \lambda c_j\]

(7)
Note that at this stage we don’t know whether the \( c_j \) indeed exist; but we do already know that, if there is an eigenfunction with a non-zero eigenvalue, then it has the above form. Substituting now (6) into (7), we obtain:

\[
\int_a^b g_j(y) \sum_{k=1}^n c_k \phi_k(y) dy = \lambda c_j \text{ for all } 1 \leq j \leq n
\]  
(8)

or

\[
\sum_{k=1}^n c_k \int_a^b g_j(y) \phi_k(y) dy = \lambda c_j \text{ for all } 1 \leq j \leq n
\]  
(9)

Finally, taking into account that we denoted \( a_{jk} = \int_a^b g_j(y) \phi_k(y) dy \), we can rewrite (9) as

\[
\begin{align*}
(a_{11} - \lambda)c_1 + a_{12}c_2 + \ldots + a_{1n}c_n &= 0 \\
(a_{21} + (a_{22} - \lambda)c_2 + \ldots + a_{2n}c_n &= 0 \\
& \vdots \\
(a_{n1}c_1 + a_{n2}c_2 + \ldots + (a_{nn} - \lambda)c_n &= 0
\end{align*}
\]

which finishes the proof of the ‘if’ direction of statement (ii). The ‘only if’ direction is now almost obvious: one simply has to note that in fact the implications at each step can be reversed. QED

Example. A linear integral operator \( A \) with kernel \( K(x, y) = x^2 + y^2 \) acts on the space of continuous functions \( L = \{ f(x), \ 0 \leq x \leq 1 \} \) in the usual way:

\[
(Af)(x) = \int_0^1 (x^2 + y^2)f(y)dy.
\]

Find all eigenvalues of this operator. Also find all eigenfunctions corresponding to non-zero eigenvalues.

**Solution.** 1. In our case

\[
K(x, y) = \phi_1(x)g_1(y) + \phi_2(x)g_2(y)
\]

where

\[
\phi_1(x) = x^2, \ g_1(y) = 1, \ \phi_2(x) = 1, \ g_2(y) = y^2.
\]

2. Hence we can find the coefficients \( a_{jk} = \int_0^1 g_j(y)\phi_k(y)dy \):

\[
a_{11} = \int_0^1 g_1(y)\phi_1(y)dy = \int_0^1 y^2dy = \frac{1}{3}, \quad a_{12} = \int_0^1 g_1(y)\phi_2(y)dy = \int_0^1 dy = 1 \\
a_{21} = \int_0^1 g_2(y)\phi_1(y)dy = \int_0^1 y^4dy = \frac{1}{5}, \quad a_{22} = \int_0^1 g_2(y)\phi_2(y)dy = \int_0^1 y^2dy = \frac{1}{3}
\]
3. To find the non-zero eigenvalues we have to solve the equation
\[
\det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = \det \begin{pmatrix} \frac{1}{3} - \lambda & 1 \\ \frac{1}{3} - \lambda & \frac{1}{5} - \lambda \end{pmatrix} = 0
\] (10)
or equivalently
\[
\left(\frac{1}{3} - \lambda\right)^2 - \frac{1}{5} = 0.
\]
The solutions to this equation are
\[
\lambda_1 = \frac{1}{3} + \frac{1}{\sqrt{5}}, \quad \lambda_2 = \frac{1}{3} - \frac{1}{\sqrt{5}}
\]
and we thus found the non-zero eigenvalues.

4. We know that the eigenfunctions related to the non-zero eigenvalues are given by
\[
f(x) = c_1 \phi_1(x) + c_2 \phi_2(x)
\]
where \((c_1, c_2)\) is any(!) non-trivial (that is non-zero) solution to
\[
(a_{11} - \lambda)c_1 + a_{12}c_2 = 0.
\]
In our case with \(\lambda = \lambda_1\) we have:
\[
-\frac{1}{\sqrt{5}}c_1 + c_2 = 0, \quad \text{or} \quad c_1 = c_2\sqrt{5},
\]
where \(c_2\) is any non-zero number. The corresponding eigenfunction is
\[
f_1(x) = c_2\sqrt{5}x^2 + c_2 = c_2(\sqrt{5}x^2 + 1).
\]
Similarly
\[
f_2(x) = \hat{c}_2(-\sqrt{5}x^2 + 1).
\]
(Note that both \(c_2\) and \(\hat{c}_2\) are arbitrary non-zero numbers which can be chosen independently from each other.)

**Exercise.** Solve the same problem for (a) \(K(x, y) = x - y, \ -1 \leq x, y \leq 1\), (b) \(K(x, y) = \sin x \cos y + \sin y \cos x, \ -\pi \leq x, y \leq \pi\).