Solutions to CW9.

Q1. 1) The series $\sum_{n=1}^{\infty}(-1)^n n^{-2}$ is converging by the alternating series test: $n^{-2} \geq (n+1)^{-2}$, and $\lim_{n\to\infty} n^{-2} = 0$.

It is absolutely convergent because $\sum_{n=1}^{\infty}\frac{1}{n^\alpha}$ converges whe $\alpha > 1$ (in our case $\alpha = 2$).

Remark. Remember that absolute convergence implie convergence. We thus proved the convergence of this series twice.

2) $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty} n^k (0.5)^n$.

Ratio test: $\lim \frac{(n+1)^k (0.5)^{n+1}}{n^k (0.5)^n} = 0.5 \lim_{n\to\infty}(1+\frac{1}{n})^k = 0.5.$

Hence this series is absolutely convergent and is also convergent.

3) Since $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$ and $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$, the serie $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test (AST in the sequel).

Howeve $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ - diverges ($\alpha < 1$).

4) $\sum_{n=1}^{\infty} \left| \frac{-2}{n^{1.5}} \right| = 2 \sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ – absolutely convergent ($\alpha > 1$). $\Rightarrow$ the series is convergent.

Q2. 1) $\lim_{n\to\infty} \left| \frac{(n+1)^2 x^{n+1}}{n^2 x^n} \right| = |x| \cdot \lim_{n\to\infty} (1+\frac{1}{n})^2 = |x|$.

Hence, by ratio test, $|x| < 1$ implies convergence, $|x| > 1$ implies divergence.

$\Rightarrow \boxed{R = 1}$ $\quad {}_{-1}(\quad)_{1}$

If $x = \pm 1$, then $\sum_{n=1}^{\infty} n^2 (\pm 1)^n$ diverges ($n^2(\pm 1)^n \not\to 0$ as $n \to \infty$). Thus

Domain of convergence of $\sum_{n=1}^{\infty} n^2 x^n$ is $(-1, 1)$.
(DoC for short).

2) As in the previous example,

$\lim_{n\to\infty} \left| \frac{(n+1)^k x^{n+1}}{n^k x^n} \right| = |x|$. and $\lim_{n\to\infty} n^k (\pm 1)^n \neq 0$

Hence DoC of $\sum_{n=1}^{\infty} n^k x^n$ is $(-1, 1)$.

3) $\lim_{n\to\infty} \left| \frac{3^{n+1} x^{n+1}}{\sqrt{n+1}} \Big/ \frac{3^n x^n}{\sqrt{n}} \right| = |x| 3 \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 3|x|$.

Hence the series converges if $3|x| < 1 \Leftrightarrow -\frac{1}{3} < x < \frac{1}{3}$. It diverges if $|x| > \frac{1}{3} \Rightarrow$

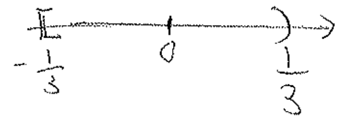
$\boxed{R = \frac{1}{3}}$

If $x = \frac{1}{3}$, then $\left(S(x) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}} x^n \right)$

$$S\left(\tfrac{1}{3}\right) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad \text{- diverges } (\alpha < 1).$$

$$S\left(-\tfrac{1}{3}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \quad \text{- converges by AST.}$$

Hence $\text{DoC} = \left[-\frac{1}{3}, \frac{1}{3}\right)$



4) $S(x) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{3^n}{n^2} x^n$;

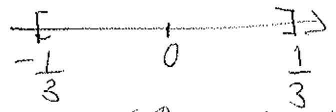
as in 3), $\lim_{n \to \infty} \left| \frac{3^{n+1} x^{n+1}}{(n+1)^2} \Big/ \frac{3^n x^n}{n^2} \right| = 3|x|$

and hence $\boxed{R = \frac{1}{3}}$. But this time

$$S\left(\tfrac{1}{3}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{- converges.}$$

$$S\left(-\tfrac{1}{3}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad \text{- converges (absolutely!)}$$

$$\text{DoC} = \left[-\frac{1}{3}, \frac{1}{3}\right]$$



5) $S(x) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} x^n$, $a_n(x) \overset{\text{def}}{=} \frac{(-1)^n x^n}{\sqrt{n+1}}$ and

$\lim_{n \to \infty} \left| \frac{a_{n+1}(x)}{a_n(x)} \right| = |x|$ (prove this!).

$\Longrightarrow \boxed{R = 1}$

$$S(1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \quad \text{converges by AST.}$$

$$S(-1) = \sum_{n=}^{\infty} \frac{(-1)^{2n}}{\sqrt{n+1}} = \sum_{n=}^{\infty} \frac{1}{\sqrt{n+1}} \quad \text{- diverges}$$

Hence DoC of $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} x^n$ is $[-1, 1]$.

6) $S(x) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{4^n\sqrt{n}} x^{2n} = \sum_{n=1}^{\infty} \frac{1}{4^n\sqrt{n}} y^n$,

where $y = x^2$. Put $a_n(y) = \frac{1}{4^n\sqrt{n}} y^n$.

Then $\lim_{n\to\infty} \left|\frac{a_{n+1}(y)}{a_n(y)}\right| = \frac{1}{4}|y| = \frac{1}{4}y$ $(y \geq 0!)$.

$0 \leq \frac{1}{4}y < 1 \iff 0 \leq \frac{1}{4}x^2 < 1 \iff |x| < 2$.

$\Rightarrow \boxed{R = 2}$.

$S(\pm 2) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is diverging.

DoC $= (-2, 2)$,

Q3. Since $|a_n| \geq 1$, $|a_n x^n| \geq |a_n|$ if $|x| \geq 1$.

But $\lim_{n\to\infty} |a_n x^n| \neq 0$ (it may not exist).

Hence $\sum_{n=1}^{\infty} a_n x^n$ diverges if $|x| \geq 1 \Rightarrow$

$R \leq 1$. ∎

Q4. Consider $\sum_{n=1}^{\infty} \underbrace{\frac{\sin nx}{n^2}}_{a_n}$ and $\sum_{n=1}^{\infty} \underbrace{\frac{1}{n^2}}_{b_n}$.

Obviously, for every x,

$$|a_n| \leq b_n.$$

But $\sum_{n=1}^{\infty} b_n$ converge $(\alpha = 2)$.

Hen $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges for all x.

and DoC $= (-\infty, \infty)$.

Q5. 1) Since $\lim_{n \to \infty} a_n = 0$, there is m_0 s.t. $\forall\ n > m_0$ we have $a_n < 1$. But then $a_n^2 < a_n$ (for $n > m_0$). Hence $\sum_{n=m_0+1}^{\infty} a_n^2 < \sum_{n=m_0+1}^{\infty} a_n$ and the series

$$\sum_{n=m_0+1}^{\infty} a_n^2 = a_{m_0+1}^2 + a_{m_0+2}^2 + \dots$$

converges by the comparison test.

Hence also $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{m_0} a_n^2 + \sum_{n=m_0+1}^{\infty} a_n^2$ converges.

2) Example. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$ diverges.