Q1

1) $\frac{n+1}{n+3}$ is strictly increasing, and tends to 1.
Thus $\inf = \frac{1}{2}$, $\sup = 1$.

2) The function $x \mapsto \frac{1}{1+x^2}$ has a maximum at $x=0$, and is symmetric with respect to 0.
$\inf = 0$ $\sup = 1$.

3) Let $a_{m,n} = \frac{1}{2^n} + \frac{1}{3^m}$. We have
$a_{m+1,n} > a_{m,n}$ and $a_{m,n+1} > a_{m,n}$
Thus $\inf = 0$ $\sup = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

~~4) Let $f(x) = x^2+3x+3$ $f'(x) = 2x+3$~~
~~Thus $f(x)$ has a minimum @ $x = -\frac{3}{2}$~~
~~and is symmetrical with respect to it.~~
~~The smallest values it assumes for $x \in \mathbb{Z}$ are $f(-1) = f(-2) = 1$~~
~~Thus $\inf = 0$ $\sup = 1$~~

4) $x^2+2x-3 < 0 \Rightarrow -3 < x < 1$
$y^2 < 9 \Rightarrow -3 < y < 3$

$-6 < x+y < 4$
inf sup

Q2

1) f is not bounded above
f is bounded below, and it attains its lower bound ($f(1) = 1$).

2) f is bounded above (by 1) and below, and it attains its bounds ($f(1) = 0$, $f(2) = 1$).

3) f is bounded above and below.

$f'(x) = \dfrac{10x}{(x^2+1)^2}$ $\quad f'(0) = 0$; $f(0) = -4$

It attains its lower bound ($f(0) = -4$), but not its upper bound (which is 1).

4) f is not bounded above and not bounded below

$\boxed{Q3}$ Assume that $a \in X$ is a greatest element, and let $b = \frac{a+r}{2}$.
Then $a < b < r$, hence $b \in X$ and $b > a$, a contradiction.

$\boxed{Q4}$ Assume a set $X \subset \mathbb{R}$ has two different greatest lower bounds, a and b. Without loss of generality, we assume that $a < b$. But then

i) a is a lower bound for X

ii) $\exists b > a$ which is a lower bound for X

And so a is not a greatest lower bound, a contradiction

$\boxed{Q5}$ Let $f: \mathbb{R} \to \mathbb{R}$ $x \mapsto x^2 - a$, where $a \in \mathbb{R}$ $a > 0$ is given.
Then $f(0) = -a < 0$. Letting $\bar{a} = \max(a, 1)$ we have
$$f(\bar{a}) = \begin{cases} a^2 - a & \text{if } a \geq 1 \\ 1 - a & \text{if } a < 1 \end{cases}$$
Then $f(\bar{a}) \geq 0$. Thus $\exists c \in [0, \bar{a}]$ such that $f(c) = 0$, that is $c^2 - a = 0$ or $c = \sqrt{a}$.

$\boxed{Q5}$ Let $a_{m,n} = \left| \frac{1}{3} - \frac{m}{2^n} \right| = \left| \frac{2^n - 3m}{3 \cdot 2^n} \right|$

Fix $n \in \mathbb{N}$, arbitrary. Then $\exists\, \bar{m} \in \mathbb{N}$ such that $|2^n - 3\bar{m}| \le 2$.

(obtained from the division of 2^n by 3:

$$2^n = 3\bar{m} + r \qquad 0 \le r < 3 \,)$$

For that value of $m = \bar{m}$ we get

$$a_{\bar{m},n} \le \frac{1}{3 \cdot 2^{n-1}}$$

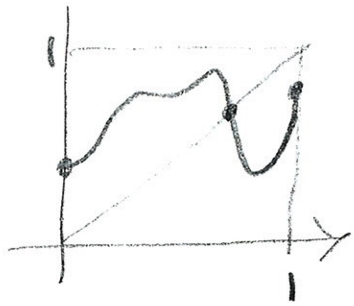
Since n was arbitrary, the result follows.

$\boxed{Q6}$ If $f(0) = 0$ or $f(1) = 1$ we have finished. So assume $f(0) > 0$ and $f(1) < 1$. Then, setting $g(x) := f(x) - x$ we have that

i) g is continuous

ii) $g(0) > 0$

iii) $g(1) < 0$



Then $\exists\, c \in [-1, 1]$ s.t. $g(c) = f(c) - c = 0$

or $f(x) = c$.