Q1

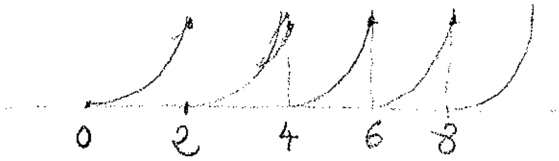
1) continuous

2) continuous

3) continuous

4) continuous

5) Discontinuous at even integers



6) Continuous $(\mp\sqrt{3} \notin \mathbb{Q})$

Q2

Roughwork: $\sqrt{x}-\sqrt{a} = (x-a)\frac{1}{\sqrt{x}+\sqrt{a}}$

$$< (x-a)\frac{1}{\sqrt{a}}$$

Proof

Let $\varepsilon > 0$ be given. Let $a > 0$ be given.

Set $\delta := \varepsilon\sqrt{a}$

Then $\forall x \in (0,\infty)$ s.t. $|x-a| < \delta$, we have

$$|f(x)-f(a)| = |\sqrt{x}-\sqrt{a}| < |x-a|\frac{1}{\sqrt{a}} < \varepsilon\frac{\sqrt{a}}{\sqrt{a}} = \varepsilon.$$

$\boxed{Q3}$ Rough work: $\dfrac{1}{x^2+1} - \dfrac{1}{a^2+1} = \dfrac{a^2+\cancel{1}-x^2-\cancel{1}}{(x^2+1)(a^2+1)}$

$$< (a-x)\underbrace{(a+x)}_{\text{must bound this}}$$

assume $|x-a|<1$. Then $|x| < 1+|a|$

and $|a+x| \leq |a|+|x| < 1+2|a| =: B(a)$

PROOF:

Let $\varepsilon>0$ be given. Let $a \in \mathbb{R}$ be given.

Set $\delta_1 := 1$

$\delta_2 := \dfrac{\varepsilon}{B(a)}$ $B(a)$ as above

$\delta := \min(\delta_1, \delta_2)$

Then $\forall x \in \mathbb{R}$ s.t. $|x-a|<\delta$, we have

$$|f(x)-f(a)| = \left|\frac{1}{x^2+1} - \frac{1}{a^2+1}\right| < |x-a|\, B(a)$$

$$< \frac{\varepsilon}{B(a)} \cdot B(a) = \varepsilon.$$

$\boxed{Q4}$ Must prove that

$\forall y \in \mathbb{R}$ $\exists \varepsilon>0$ s.t. $\forall \delta>0$ $\exists x$ s.t. $|x-0|<\delta$ AND $|f(x)-y| \geq \varepsilon$

Let $y \in \mathbb{R}$ be given.

Set $\varepsilon := 1$ (or indeed any +ve number ≤ 1).

Let $\delta>0$ be given

set $x := -\delta/2$ if $y \geq 0$

$x := \delta/2$ if $y<0$

Then we have $|x|<\delta$ and

if $y \geq 0$ $|f(x)-y| = |-1-y| \geq 1 \geq \varepsilon$

if $y<0$ $|f(x)-y| = |1-y| > 1 \geq \varepsilon$

$\boxed{Q5}$ 1) Must prove that

$$\sin(x)-\sin(a) = 2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right) \qquad (*)$$

This is derived from the addition formula

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

which gives

$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta).$$

Letting $x := \alpha+\beta \quad a := \alpha-\beta$

we get $\alpha = \frac{x+a}{2} \quad \beta = \frac{x-a}{2}$

and $(*)$ follows.

2) From $(*)$ and the given estimate, we get

$$|\sin(x)-\sin(a)| = 2\left|\cos\left(\frac{x+a}{2}\right)\right|\left|\sin\frac{x-a}{2}\right|$$

$$\le \cancel{2}\cdot 1 \cdot \frac{1}{\cancel{2}}|x-a| = |x-a|$$

<u>Proof</u> of continuity of sine function

Let $\varepsilon>0$ be given. Let $a\in\mathbb{R}$ be given.

Set $\delta := \varepsilon$.

Then $\forall x\in\mathbb{R}$

$$|x-a|<\delta \Rightarrow |\sin(x)-\sin(a)| \le |x-a|$$

$$< \delta = \varepsilon.$$