Q1 1) $S_n := \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$.

The largest summand is $\frac{1}{n+1}$;
The smallest summand is $\frac{1}{2n}$. There are n summands,
Thus
$$\frac{1}{2} = \frac{n}{2n} \le S_n < \frac{n}{n+1} < 1$$

2) (Later)

Q2 Later

Q3 Write each statement as $P(n) := LHS(n) = RHS(n)$.

1) $LHS(1) = 1! \cdot 1 = 1 \quad RHS(1) = 2! - 1 = 1$.
Assume $P(n)$ is true for some $n \ge 1$. Then
$LHS(n+1) = (n+1)! - 1 + (n+1)!(n+1) =$
$= (n+1)!(n+2) - 1 = (n+2)! - 1 = RHS(n+1)$.

2) $LHS(1) = \frac{1}{2} \quad RHS(1) = 6 - \frac{11}{2} = \frac{1}{2}$
Assume $P(n)$ is true for some $n \ge 1$. Then

$LHS(n+1) = 6 - \frac{1}{2^n}(n^2+4n+6) + \frac{(n+1)^2}{2^{n+1}}$

$= 6 - \frac{1}{2^{n+1}}(2n^2+8n+12-n^2-2n-1)$

$= 6 - \frac{1}{2^{n+1}}((n+1)^2+4(n+1)+6) = LHS(n+1)$

3) $LHS(2) = \frac{1}{3}$ $RHS(2) = \frac{3}{4} - \frac{5}{12} = \frac{1}{3}$

Assume $S(n)$ for some $n \geq 2$.

$$LHS(n+1) = \frac{3}{4} - \frac{2n+1}{2n(n+1)} + \frac{1}{(n+1)^2-1} =$$

$$= \frac{3}{4} - \frac{(n+2)(2n+1) - 2(n+1)}{2n(n+1)(n+2)}$$

$$= \frac{3}{4} - \frac{2n^2+3n}{2n(n+1)(n+2)} = \frac{2(n+1)+1}{2(n+1)(n+2)} = RHS(n)$$

4) $LHS(1) = \frac{1}{p(p+q)} = RHS(1)$

Assume $S(n)$ true for some $n \geq 1$.

$$LHS(n+1) = \frac{n}{p(p+nq)} + \frac{1}{(p+nq)(p+(n+1)q)}$$

$$= \frac{n(p+(n+1)q)+p}{p(p+nq)(p+(n+1)q)}$$

$$= \frac{(n+1)p + n(n+1)q}{p(p+nq)(p+(n+1)q)} = \frac{(n+1)\cancel{(p+nq)}}{p\cancel{(p+nq)}(p+(n+1)q)} = LHS(n+1)$$

$\boxed{Q2}$ $\quad \mathcal{P}(n) := (2^n \geq n^2)$

$\mathcal{P}(4)$: $\quad 2^4 \geq 4^2$ is true

Assume $\mathcal{P}(n)$ true for some $n \geq 4 =: m$. Then

$2^{n+1} = 2 \cdot 2^n \geq 2n^2$

Now $(n+1)^2 = n^2\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$

For $n \geq 4$ $\quad 1 + \frac{2}{n} + \frac{1}{n^2} \leq 1 + \frac{1}{2} + \frac{1}{16} < 2$

Thus, for $n \geq 4$ $\quad 2^{n+1} \geq (n+1)^2$ completing the induct.

Note: $\mathcal{P}(3)$ is false so $m = 4$ is sharp

$\mathcal{P}(2)$ is true, which may lead to incorrect statements.

$\boxed{Q1}$ 2) $\quad n! = \prod_{k=1}^{n} k = \prod_{k=2}^{n} k$

Smallest term in product is 2, largest is n.
There are $n-1$ terms, hence

$$2^{n-1} \leq n! \leq n^{n-1}$$

$\boxed{Q4}$

1) True: Let $\frac{n}{3n-1} = \frac{1}{3} + \varepsilon_n$

$$\varepsilon_n = \frac{n}{3n-1} - \frac{1}{3} = \frac{3n-3n+1}{3(3n-1)}$$

we require $\left| \cancel{\frac{1}{3}} - \cancel{\frac{1}{3}} + \varepsilon_n \right| = |\varepsilon_n| = \varepsilon_n < \frac{1}{10^2}$

that is, $3(3n-1) > 100$, or $n \geq 12 =: m$

2) False: $\forall m > 0 \ \exists n \geq m$ s.t $\left| \frac{1}{3} - \frac{1}{3n+1} \right| \geq 10^{-2}$

Let m be given.

Let $\varepsilon_n := \frac{1}{3} - \frac{1}{3n+1} = \frac{3n-2}{3(3n-1)} > 0 \quad \forall n \in \mathbb{N}$

$|\varepsilon_n| \geq \frac{1}{100} \iff \varepsilon_n \geq \frac{1}{100} \iff 100(3n-2) \geq 3(3n-1)$

$\iff 291n \geq 203 \iff n \geq \frac{203}{291}$

This condition is always verified, since $n \in \mathbb{N}$.

Thus we set $n := m$.

3) From problem 2 part 4 we have

$$S_n := \sum_{r=1}^{n} \frac{1}{5r-4}\cdot\frac{1}{5r+1} = \sum_{r=1}^{n} \frac{1}{[1+(r-1)\cdot 5](1+r\cdot 5)} = \frac{n}{1+5n}$$

Now $\frac{n}{1+5n} < \frac{1}{5}$, hence

$$\left|S_n - \frac{1}{5}\right| = \frac{1}{5} - S_n = \frac{1}{5} - \frac{n}{1+5n} = \frac{1}{5(1+5n)}$$

Thus we require

$$0 < \frac{1}{5(1+5n)} < \frac{1}{1000} \iff 199 < 5n$$

$$\iff n > \frac{199}{5} \quad \text{or} \quad n \geq 40 =: m$$

4) True. We make use of the following

$$\forall n \in \mathbb{N} \quad \sqrt{n+1} > \sqrt{n}$$

$$\forall x \in \mathbb{R},\ x > 0 \quad \sqrt{1+x} < 1+x$$

$$\left|\sqrt{n+1} - \sqrt{n}\right| = \sqrt{n}\left(\sqrt{1+\frac{1}{n}} - 1\right) < \sqrt{n}\left(1 + \frac{1}{n} - 1\right) = \frac{1}{\sqrt{n}}$$

Thus

$$\left|\sqrt{n+1} - \sqrt{n}\right| < \frac{1}{10^{40}} \Leftarrow \frac{1}{\sqrt{n}} < \frac{1}{10^{40}}$$

$$\iff n > 10^{20}$$

Let $m := 10^{20} + 1$.

$\boxed{Q5}$ $x_1 > 0$ and $x_n > 0 \Rightarrow x_{n+1} > 0$,

so all terms are positive.

$x_2 = \frac{1}{4} < 1$.

$0 < x_n < 1 \Rightarrow x_{n+1} < \frac{x_n \cdot 1}{2 \cdot 0 + 2} < x_n$

Iterating the inequality

$$x_{n+1} \leq \frac{x_n^2}{4} \qquad n \in \mathbb{N}$$

we get

$$x_2 \leq \frac{1}{2^2} \qquad x_3 \leq \frac{1}{2^{4+2}} \quad x_4 \leq \frac{1}{2^{12+2}}.$$

Writing

$$x_n \leq \frac{1}{2^{e_n}}$$

we find

$$e_1 = 0 \qquad e_{n+1} = 2e_n + 2, \quad n \in \mathbb{N}$$

that is

$$e_n = \sum_{k=1}^{n-1} 2^k = 2^n - 2.$$

Thus

$$x_n \leq \frac{1}{2^{2^n - 2}}.$$

We must have $2^{2^n-2} > 10^{300}$

Because $2^{10} > 10^3$, then $2^{1000} > 10^{300}$

So we require $2^n - 2 > 1000$

or $n \geq 10$ $\quad$ $m = 10$