Q1

1. False: $\exists x \in \mathbb{R}$ s.t. $x^2 - 2x + 1 \leq 0$.
   Set $x := 1$. Then $1^2 - 2 \cdot 1 + 1 = 0$, as desired.

2. True: $4x^2 + 8x < -3 \iff 4x^2 + 8x + 3 < 0 \iff 4\left(x + \frac{1}{2}\right)\left(x + \frac{3}{2}\right) < 0$
   The roots of the LHS are $-\frac{1}{2}$, $-\frac{3}{2}$ and the leading coeff is +ve. So $x := -1$ is the desired integer.

3. True: Let $y \in \mathbb{R}$ be given. Then $(-y)^{2n} = [(-y)^n]^2 \geq 0$, as desired.

4. True: Let $x, y, z \in \mathbb{R}$ be given. Then $(x-y)^2 + (y-z)^2 + (x-z)^2 \geq 0$
   $\Rightarrow 2(x^2 + y^2 + z^2) - 2(xy + xz + yz) \geq 0 \Rightarrow x^2 + y^2 + z^2 \geq xy + xz + yz$

5. True. Let $x \in \mathbb{R}$ be given. Then $x > 1 \Rightarrow x > 0$,
   and therefore $x > 1 \Rightarrow x^2 = x \cdot x > x \cdot 1 = x$.

6. False: $\exists x, y \in \mathbb{R}$ s.t. $x < y$ and $x^2 \geq y^2$.
   Set $x := -1$ and $y := 0$. $-1 < 0$ and $(-1)^2 = 1 \geq 0 = 0^2$.

7. True: Let $x \in \mathbb{R}$ be given. Then $x^2 < 9 \Rightarrow x < 3$
   and since $3 < 4$, $x < 4$.

8. False: $\forall z \in \mathbb{R}$ $\exists x \in \mathbb{R}$ s.t. $x^2 + zx - 2 \leq 0$
   Let $z \in \mathbb{R}$ be given.
   Set $x := 0$. Then $0^2 + z \cdot 0 - 2 = -2 \leq 0$, as desired

9. True: Let $x, y, z, w \in \mathbb{R}$ be given.

Then $wz^2 + xz + y = 0$ implies that the discriminant Δ w.r.t. z is non-negative.

$\Delta = x^2 - 4yw \geq 0 \implies x^2 \geq 4yw$, as desired

10. False: $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}$ $x^3 \neq y^2 - 10^2$

(Note $y^2 - 10^2 \geq -100$)

set $x := -5$

Let $y \in \mathbb{R}$ be given

Then $x^3 = -125 < -100 \leq y^2 - 10^2$, as desired.

11. True: $\sqrt{5} < x < \sqrt{6} \iff 5 < x^2 < 6$ and $x > 0$

$\iff 45 = 3^2 \cdot 5 < (3x)^2 < 3^2 \cdot 6 = 54$ and $x > 0$.

Set $x := \frac{7}{3}$ then $45 < 7^2 < 54$ and $\frac{7}{3} > 0$.

12. True: set $a := 3$, $b := 4$, $c := 5$. Then $3^2 + 4^2 = 25 = 5^2$.

13. False: $\exists n \in \mathbb{N}$ s.t. $n^2 + n + 41$ is not prime.

Set $n := 41$.

Then $41^2 + 41 + 41 = 41(41 + 1 + 1) = 41 \cdot 43$, as desired.

[Note: $n^2 + n + 41$ is prime for $n = 1, 2, \ldots, 39$ (Euler);

for $n = 40$, $40^2 + 40 + 41 = 40^2 + 2 \times 40 + 1 = (40+1)^2 = 41^2$]

Q2

1. $|x+y+z| = |(x+y)+z| \le |x+y| + |z| \le |x|+|y|+|z|$

2. $||x|-|y|| = |x|-|y|$ or $|y|-|x|$. We have $|x| = |x-y+y|$
$\le |x-y|+|y| \Rightarrow |x|-|y| \le |x-y|$. Also $|y| = |x-(x-y)|$
$\le |x| + |x-y| \Rightarrow |y|-|x| \le |x-y|$. Hence $||x|-|y|| \le |x-y|$.

3. $|x-y| < z \iff x-y < z$ and $-(x-y) < z$
$\iff x < y+z$ and $y-z < x \iff y-z < x < y+z$.

Q3

(a) $P(2) \Rightarrow a_1 a_2 \le \left(\frac{a_1+a_2}{2}\right)^2, \; a_3 a_4 \le \left(\frac{a_3+a_4}{2}\right)^2$

$$\Rightarrow a_1 a_2 a_3 a_4 \le \left(\frac{a_1+a_2}{2}\right)^2 \left(\frac{a_3+a_4}{2}\right)^2 \qquad (*)$$

$P(2) \Rightarrow \frac{a_1+a_2}{2} \, \frac{a_3+a_4}{2} \le \left(\frac{a_1+a_2+a_3+a_4}{2^2}\right)^2$

$$\Rightarrow \left(\frac{a_1+a_2}{2}\right)^2 \left(\frac{a_3+a_4}{2}\right)^2 \le \left(\frac{a_1+a_2+a_3+a_4}{2^2}\right)^4 \qquad (**)$$

$(*)$ and $(**)$ give the desired result.

(b) $P(4) \Rightarrow a_1 a_2 a_3 \left(\frac{a_1+a_2+a_3}{3}\right) \le \left(\frac{a_1+a_2+a_3+(a_1+a_2+a_3)/3}{4}\right)^4$

$$\Rightarrow a_1 a_2 a_3 \left(\frac{a_1+a_2+a_3}{3}\right) \le \left(\frac{a_1+a_2+a_3}{3}\right)^4$$

since the a_i are positive, the result follows.

(c) Set $\pi := a_1 \cdot a_2 \cdot a_3 \cdot a_4$ $\quad \sigma := a_1 + a_2 + a_3 + a_4$

$\pi' := a_5 \cdot a_6\, a_7\, a_8$ $\quad \sigma' := a_5 + a_6 + a_7 + a_8$

$$\mathcal{P}(4) \Rightarrow \quad \pi \le \left(\frac{\sigma}{4}\right)^4, \quad \pi' \le \left(\frac{\sigma'}{4}\right)^4$$

$$\Rightarrow \pi\pi' \le \left(\frac{\sigma}{4}\right)^4 \left(\frac{\sigma'}{4}\right)^4 \quad (*)$$

$$\mathcal{P}(2) \Rightarrow \quad \frac{\sigma}{4} \cdot \frac{\sigma'}{4} \le \left(\frac{\sigma + \sigma'}{8}\right)^2 \quad (**)$$

$(*)$ and $(**)$ give $\mathcal{P}(8)$

Now set $\pi = a_1 \cdot a_2\, a_3\, a_4\, a_5$ $\quad \sigma = a_1 + a_2 + a_3 + a_4 + a_5$

$$\mathcal{P}(8) \Rightarrow \quad \pi \cdot \left(\frac{\sigma_5}{5}\right)^3 \le \left(\frac{\sigma_5 + \frac{3}{5}\sigma_5}{8}\right)^8 = \left(\frac{\cancel{8}\,\sigma_5}{\cancel{8}\,5}\right)^8$$

Because $\frac{\sigma}{5} > 0$, the result follows.