MAS/111 Convergence and Continuity: Coursework 9

DEADLINE: Thursday of week 12, at 11:00 am.

Problem 1. Explain, with a reason, which of the following series are convergent, absolutely convergent, convergent but not absolutely convergent?

1) $\sum_{n=1}^{\infty}(-1)^n n^{-2}$ 2) $\sum_{n=1}^{\infty}(-1)^{n-1} n^k \, (0.5)^n$, where $k \in \mathbb{N}$

3) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ 4) $\sum_{n=1}^{\infty} \frac{-2}{n^{1.5}}$

Problem 2. Find the radius of convergence, and the domain of convergence, for each of the following power series

1) $\sum_{n=1}^{\infty} n^2 \, x^n$ 2) $\sum_{n=1}^{\infty} n^k \, x^n$, where $k \in \mathbb{N}$

3) $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}} \, x^n$ 4) $\sum_{n=1}^{\infty} \frac{3^n}{n^2} \, x^n$

5) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \, x^n$ 6) $\sum_{n=1}^{\infty} \frac{1}{4^n \sqrt{n}} \, x^{2n}$

[In 6), let $y = x^2$.]

Problem 3. Given that the coefficients of the power series $\sum_{n=1}^{\infty} a_n x^n$ are non-zero integers, show that the radius of convergence is at most 1.

[Use the fact that $|a_n| \geq 1$ and check whether the necessary condition for convergence of a series holds when $|x| \geq 1$.]

Problem 4. We say that the domain of convergence of a series $\sum_{n=1}^{\infty} f_n(x)$ is the set of those $x \in \mathbb{R}$ for which this series converges. Consider the following example of such a series: $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$. What is the domain of convergence of this series? Explain your answer.

Problem 5. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms.

1) Show that the series $\sum_{n=1}^{\infty} a_n^2$ converges.

2) Show that the series $\sum_{n=1}^{\infty} \sqrt{a_n}$ does not necessarily converge.

[In 1) use two facts: (i) a_n converges to 0 as $n \to \infty$ and (ii) for sufficiently large n, $a_n^2 \leq a_n$; then apply the comparison test. In 2) find an example which demonstrates the statement.]