MAS/111 Convergence and Continuity: Coursework 8

DEADLINE: Thursday of week 11, at 11:00 am.

Problem 1. Real numbers can be defined via their decimal expansions, such as

$$0.3192\ldots = 3 \times \frac{1}{10^1} + 1 \times \frac{1}{10^2} + 9 \times \frac{1}{10^3} + 2 \times \frac{1}{10^4} + \cdots$$

These expansions are examples of infinite series.

1) Find the rational numbers presented by the following sums:

$$\sum_{k=1}^{\infty} \frac{2}{10^k}; \quad \frac{1}{10} + \frac{2}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \ldots = \sum_{k=1}^{\infty} (\frac{1}{10^{2k-1}} + \frac{2}{10^{2k}})$$

2) Determine the rational number having the following decimal expansion

$$0.2\ 035\ 035\ 035\ 035\ldots$$

Problem 2. The series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad s > 1 \tag{1}$$

converges if $s > 1$ and diverges if $0 \leq s \leq 1$.

Consider also another series:

$$S := \sum_{n=1}^{\infty} a_n \tag{2}$$

The following statements are useful for determining whether or not a series is converging.

1. The necessary and sufficient condition for convergence of an alternating series (see notes).

2. The NECESSARY (but NOT sufficient) condition for convergence of a series: $\lim_{n\to\infty} a_n = 0$

3. The comparison test: consider one more series

$$\sum_{n=1}^{\infty} b_n \tag{3}$$

If for all n $0 \leq a_n \leq b_n$ then: (a) series (2) converges if series (3) converges; (b) series (3) diverges if series (2) diverges.

4. The ratio test: suppose that $\forall n \geq 1$ $a_n \neq 0$ and that $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lambda$ exists. Then: (a) series (2) converges if $\lambda < 1$; (b) series (2) diverges if $\lambda > 1$. Test is inconclusive if $\lambda = 1$.

Determine, with a reason, whether each of the following series is convergent or divergent. To do that you may/should use the information about series (1) as well as statements 1 - 4.

1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^{\frac{1}{3}}}$ 2) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2+n+1}$

3) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^8}$ 4) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

5) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n(n+2)}}$ 6) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n^2+1)}}$

7) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+2)}}$ 8) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{\sqrt{n^2+1}}$

Problem 3. The condensation test: Let a_n be a *decreasing* sequence of positive real numbers. Then

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \qquad \Longleftrightarrow \qquad \sum_{m=0}^{\infty} 2^m a_{2^m} \quad \text{converges.}$$

For the following series, use the condensation/comparison test to determine convergence/divergence.

1) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ 2) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$

3) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$ 4) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln(n)\ln(\ln(n))}$

[You may assume the following properties of the logarithm: *i)* for all $a, b > 0$, $\ln(ab) = \ln(a)+\ln(b)$ (in particular $\ln(a^n) = n\ln(a)$); *ii)* $\ln(1) = 0 < \ln(2) < 1$; *iii)* $a < b \Longrightarrow \ln(a) < \ln(b)$.]