

## MAS/111 Convergence and Continuity: Coursework 5

*DEADLINE: Thursday of week 8, at 11:00 am.*

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**Problem 1.** In each case below, state whether or not the function  $f : X \rightarrow \mathbb{R}$  is continuous. If not, write down the elements of  $X$  where it is discontinuous.

$X$	$x \mapsto f(x)$
1) $\mathbb{R}$	$x \mapsto \begin{cases} x & \text{if } x \leq 1 \\ \frac{1}{x} & \text{if } x > 1 \end{cases}$
2) $(-3, 3)$	$x \mapsto \frac{1}{x+3}$
3) $\mathbb{R} \setminus \{1, -1\}$	$x \mapsto \frac{x}{x^2 - 1}$
4) $\mathbb{R}$	$x \mapsto (x - 2n)^2$ if $2n - 1 \leq x < 2n + 1$ , $n \in \mathbb{Z}$
5) $\mathbb{R}$	$x \mapsto (x - 2n)^2$ if $2n \leq x < 2n + 2$ , $n \in \mathbb{Z}$
6) $\mathbb{Q}$	$x \mapsto \begin{cases} 0 & \text{if } x^2 \leq 3 \\ 1 & \text{if } x^2 > 3. \end{cases}$

**Problem 2.** Prove from first principles that the function  $x \mapsto \sqrt{x}$  is continuous in  $\mathbb{R}^+ = (0, \infty)$ .

**Problem 3.** Prove from first principles that the function

$$f : \mathbb{R} \rightarrow \mathbb{R} \quad x \mapsto \frac{1}{x^2 + 1}$$

is continuous.

**Problem 4.** Let  $y$  be a real number, and let

$$f : \mathbb{R} \rightarrow \mathbb{R} \quad x \mapsto \begin{cases} -1 & x < 0 \\ y & x = 0 \\ 1 & x > 0 \end{cases}$$

Prove that  $f$  is discontinuous at  $x = 0$  for any choice of  $y$ .

[Write down carefully the quantified statement to be proved.]

**Problem 5.** Let  $x, y \in \mathbb{R}$ .

1) From the addition formula for the sine function,<sup>1</sup> prove that

$$\sin(x) - \sin(y) = 2 \sin\left(\frac{x-y}{2}\right) \cos\left(\frac{x+y}{2}\right).$$

2) Using the above formula, and assuming the validity of the estimate

$$0 \leq |\sin(x)| \leq |x|$$

prove from first principles that the function  $x \mapsto \sin(x)$  is continuous in  $\mathbb{R}$ .

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<sup>1</sup>Look it up, if you do not remember it.