MAS/111 Convergence and Continuity: Coursework 2

DEADLINE: Thursday of week 4, at 11:00 am.

Problem 1. Consider the following statements

1. $\forall x \in \mathbb{R},\ x^2 - 2x + 1 > 0$
2. $\exists x \in \mathbb{Z},\ 4x^2 + 8x < -3$
3. $\forall y \in \mathbb{R}, \forall n \in \mathbb{N}, (-y)^{2n} \geq 0$
4. $\forall x, y, z \in \mathbb{R}, xy + xz + yz \leq x^2 + y^2 + z^2$
5. $\forall x \in \mathbb{R},\ (x > 1 \Longrightarrow x^2 > x)$
6. $\forall x, y \in \mathbb{R},\ (x < y \Longrightarrow x^2 < y^2)$
7. $\forall x \in \mathbb{R}, (x^2 < 9 \Longrightarrow x < 4)$
8. $\exists z \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}, x^2 + zx - 2 > 0$
9. $\forall x, y, z, w \in \mathbb{R}, (wz^2 + xz + y = 0 \Longrightarrow x^2 \geq 4wy)$
10. $\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}$ s.t. $x^3 = y^2 - 10^2$
11. $\exists x \in \mathbb{Q}$ s.t. $\sqrt{5} < x < \sqrt{6}$
12. $\exists a, b, c \in \mathbb{N}$ s.t. $a^2 + b^2 = c^2$
13. $\forall n \in \mathbb{N},\ n^2 + n + 41$ is prime.

If a statement is true, prove it; if it is false, write down explicitly its negation, and then prove it.

Problem 2. Prove that, for all $x, y, z \in \mathbb{R}$

1. $|x + y + z| \leq |x| + |y| + |z|$
2. $||x| - |y|| \leq |x - y|$
3. $|x - y| < z \iff y - z < x < y + z$

Problem 3. For $n \in \mathbb{N}$, let $\mathcal{P}(n)$ denote the statement

$$\forall a_i \in \mathbb{R}, a_i > 0,\ 1 \leq i \leq n \qquad \prod_{i=1}^{n} a_i \leq \left(\frac{1}{n} \sum_{i=1}^{n} a_i \right)^n .$$

The validity of $\mathcal{P}(1)$ is immediate; $\mathcal{P}(2)$ is proved in the web-book.

1) Prove $\mathcal{P}(4)$. [Apply $\mathcal{P}(2)$ three times, to the pairs (a_1, a_2), (a_3, a_4), and $((a_1 + a_2)/2, (a_3 + a_4)/2)$.]

2) Prove $\mathcal{P}(3)$. [Use $\mathcal{P}(4)$.]

3) Prove $\mathcal{P}(5)$. [Prove $\mathcal{P}(8)$ first.]