MAS/111 Convergence and Continuity: Coursework 1

DEADLINE: Thursday of week 3, at 11:00 am.

Problem 1. The following statements hold for all real numbers x, y

1. $x0 = 0x = 0$
2. $x(-y) = (-x)y = -xy$
3. $(-x)(-y) = xy$
4. $(x-y)^2 = x^2 - 2xy + y^2$
5. $x > 0 \iff -x < 0$
6. $1 > 0$
7. $x^2 \geq 0$
8. $x \neq 0 \implies x$ has only one multiplicative inverse
9. $0 < x < y \implies 0 < \dfrac{1}{y} < \dfrac{1}{x}$.

Prove them directly from the axioms for $\mathbb{R}$. Indicate *precisely* which axioms are used at *each* stage of the proofs.

Problem 2. Prove that if x and y are positive real numbers, then

$$x < y \iff x^2 < y^2.$$

Prove that if x and y are arbitrary real numbers, the statement is false.

Problem 3. Let $i, n \in \mathbb{Z}$ and $n > 0$. Prove that

$$\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}. \tag{1}$$

You may assume that, if $i < 0$ or $i > n$, then $\binom{n}{i} = 0$.

Problem 4. Prove by induction that if n is a positive integer and $x, y \in \mathbb{R}$, then

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

Use (1) in the inductive step.

Problem 5. Prove by induction that if x is a real number such that $x > -1$, then

$$(1+x)^n \geq 1 + nx$$

for all positive integers n.

Problem 6. Prove by induction that for each positive integer n

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k \right)^2.$$