The Ratio Test

Remember that $\lim_{n\to\infty} v_n = v$ if for any $\varepsilon > 0$ there is $m \in \mathbb{N}$ s. t. for all $n \geq m$

$$v - \varepsilon < v_n < v + \varepsilon.$$

Remark. The last inequality is equivalent to $|v - v_n| < \varepsilon$ (prove this). Also, saying 'for all $n > m$' is equivalent to saying 'for all $n \geq m$'.

Theorem 0.1
Consider a series

$$S = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ... \tag{1}$$

and suppose that $a_n \neq 0$, $n = 1, 2, ...$ and that the following limit exists:

$$\lambda = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \tag{2}$$

Then
(i) Series (1) is absolutely convergent if $\lambda < 1$;
(ii) Series (1) diverges if $\lambda > 1$
The test is inconclusive if $\lambda = 1$.

Proof. The existence of limit (2) implies that for any $\varepsilon > 0$ there is an $m \in \mathbb{N}$ such that $\forall\, n \geq m$

$$\lambda - \varepsilon < \left| \frac{a_{n+1}}{a_n} \right| < \lambda + \varepsilon. \tag{3}$$

Proofs of (i) and (ii) make use of this inequality.
(i) Since $\lambda < 1$ we can choose $\varepsilon > 0$ so that $q := \lambda + \varepsilon < 1$. Then for appropriate m and $n \geq m$ we have: $|\, a_{n+1}| < (\lambda + \varepsilon)\, |\, a_n| = q\, |\, a_n|$. Hence

$$|\, a_{m+1}| < q\, |\, a_m|\,, \quad |\, a_{m+2}| < q\, |\, a_{m+1}| < q^2\, |\, a_m|\,, \quad ..., \quad |\, a_n| < q^{n-m}\, |\, a_m|\,.$$

(in fact this is simple induction for $n > m$). Define a new series with terms b_n given by

$$b_n = |\, a_n| \quad \text{if} \quad n < m \quad \text{and} \quad b_n = q^{n-m}\, |\, a_m| \quad \text{if} \quad n \geq m.$$

Obviously, $|\, a_n| \leq b_n$ for $n \geq 1$. This series converges because

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{m-1} |\, a_n| + \sum_{n=m}^{\infty} |\, a_m|\, q^{n-m} = \sum_{n=1}^{m-1} |\, a_n| + \frac{|\, a_m|}{1-q}.$$

and therefore series (1) is absolutely convergent (by comparison test) and thus also convergent. □
(ii) Since $\lambda > 1$ we can choose $\varepsilon > 0$ so that $\lambda - \varepsilon > 1$. The existence of limit (2) implies that there is $m \in \mathbb{N}$ such that (3) holds $\forall\, n \geq m$. But then for $n \geq m$ we have $|\, a_{n+1}| > (\lambda - \varepsilon)\, |\, a_n| \geq |\, a_n|$. This implies that $\lim_{n\to\infty} |\, a_n| \neq 0$ and therefore series (1) cannot converge. □
Remark. In fact it is easy to see that if $\lambda > 1$ then $|\, a_n| > (\lambda - \varepsilon)^{n-m}\, |\, a_m| \to \infty$ as $n \to \infty$.