MAS111 Convergence and Continuity
Selected Important Topics

The purpose of this list of questions is to help you to prepare for the exam by revising the material of the course in a systematic way. Apart of the ability to provide answers to these questions you are supposed to be able to solve problems similar to those considered in courseworks.

---

1. Basics
   (a) Prove that $1 > 0$.
   (b) Prove that if $x \geq -1$ then $(1+x)^n \geq 1 + nx$, where $n \geq 1$ is an integer number. (This is the so called Bernoulli inequality.)
   (c) What is the definition of a limit of a sequence (a_n)? Prove the following properties of limits:
   Theorem. *Suppose that* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *Then*
   (1) $\lim_{n\to\infty} ca_n = ca$, *where* c *is a constant*
   (2) $\lim_{n\to\infty}(a_n + b_n) = a + b$
   (3) $\lim_{n\to\infty}(a_n b_n) = ab$
   If, in addition to the above conditions, $b \neq 0$ *then*
   (4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.
   (d) What is the definition of the least upper bound of a subset A of the set of real numbers $\mathbb{R}$?
   (e) State (do not prove) the theorem about the existence of the least upper bound of a set.
   (f) State and prove the Bolzano-Weierstrass Theorem.

2. The existence of $e \stackrel{\text{def}}{=} \lim_{n\to\infty}(1 + \frac{1}{n})^n$.
   Define a sequence of real numbers by $a_n = (1 + \frac{1}{n})^n,\ n \geq 1$.
   (a) Prove that $a_n \geq 2$
   Hint. You can either use the binomial theorem or the Bernoulli inequality.
   (b) Prove that a_n is a monotone sequence of numbers, namely, that $a_{n+1} \geq a_n$.
   (c) Prove that $a_n \leq 3$. Conclude from here and the previous statement that the limit $e \stackrel{\text{def}}{=} \lim_{n\to\infty}(1 + \frac{1}{n})^n$ exists and that $2 \leq e \leq 3$.

3. Properties of continuous functions.

(a) Define what is meant by saying that a function $f(x)$, $f : X \mapsto \mathbb{R}$ is continuous at $a \in X$. Here X is a subset of the set of real numbers $\mathbb{R}$.

    And what is meant by saying that a function $f : X \mapsto \mathbb{R}$ is continuous on X.

(b) Prove that prove that if $f(x) = c$ for all $x \in X$ then f is a continuous on X.

(c) Prove that if $f : X \mapsto \mathbb{R}$ and $g : X \mapsto \mathbb{R}$ are continuous functions on X then

    (a) $f(x) + g(x)$ is a continuous function,

    (b) $f(x)g(x)$ is a continuous function.

    (c) $f(x)/g(x)$ is a continuous function at each point $x \in X$ where $g(x) \neq 0$.

(d) Suppose that if $f : [a, b] \mapsto [c, d]$ is a continuous function on $[a, b]$ taking values in $[c, d]$ and $g : [c, d] \mapsto \mathbb{R}$ then the composite function $g(f(x)) : [a, b] \mapsto \mathbb{R}$ is continuous.

4. Properties of continuous functions and some of their applications.

    (a) State and prove the intermediate value theorem.

    (b) Prove that for every real number $a > 0$ there is a root k of a, that is there is a real positive number c such that $c^k = a$. Here $k \in \mathbb{N}$.

    (c) Let $f(x)$ be a continuous function on $[a, b]$, where a and b are finite numbers. State and prove the properties of continuous functions concerned with the maximal and minimal values of such functions.

5. The Cauchy criterion.

    (a) State the Cauchy criterion for convergence of a sequence.

    (b) Prove the simple part of the Cauchy criterion for convergence of a sequence: if $\lim_{n\to\infty} a_n = \ell$ then for any $\varepsilon > 0$ there is N such that for any $n > N$ and any $m > N$ the inequality $|a_n - a_m| < \varepsilon$ holds true.

    (c) State and prove the Cauchy criterion for convergence of a sequence.

6. Series.

    (a) Consider a series $S = \sum_{n=1}^{\infty} a_n$.

        What is the definition of partial sums of a series S?

        What is meant by saying that a series S converges?

    (b) What is an alternating series? State and prove the necessary and sufficient conditions for convergence of an alternating series with decaying elements.

    (c) Prove that if a series $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

        Give an example of a diverging series whose elements satisfy the property $\lim_{n\to\infty} a_n = 0$.

    (d) What is the definition of an absolutely convergent series? Prove that an absolutely convergent series is convergent.

        Hint. You are supposed to use, without proof, the Cauchy criterion for convergence of a sequence.

(e) State and prove the comparison test for convergence of a series.

(f) State and prove the condensation test.

(g) State (do not prove) the condensation test. Using the condensation test prove that

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is a diverging series.

(h) State and prove the ratio test for the convergence of a series.

(i) What is the definition of a power series?

What is the definition of the domain of convergence of a power series?

What is the definition of the radius of convergence of a power series?

(j) What is the domain of convergence of the following series:

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \ ?$$

What is the domain of convergence of this series?