Convergence of Alternating Series

Definition. A series

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + ... + (-1)^{n-1} a_n + ...$$

where $a_n \geq 0$, $n = 1, 2, ...$, is called an alternating series.

Theorem 0.1
Suppose that $\forall j \geq 1$ *the inequality* $a_j \geq a_{j+1}$ *holds. Then the alternating series converges if and only if* $\lim_{n\to\infty} a_n = 0$.

Proof. The partial sums $S_n \stackrel{\text{def}}{=} \sum_{k=1}^{n} (-1)^{k-1} a_k$ have the following properties:
1. All partial sums are non-negative: $S_1 = a_1 \geq 0$ and for $n \geq 1$

$$S_{2n} = \sum_{k=1}^{2n} (-1)^{k-1} a_k = \sum_{k=1}^{n} (a_{2k-1} - a_{2k}) \geq 0 \quad \text{and} \quad S_{2n+1} = S_{2n} + a_{2n+1} \geq 0.$$

2. $S_n \leq a_1$ because $S_n = a_1 - (a_2 - a_3 + ... + (-1)^{n-2} a_n)$ and $a_2 - a_3 + ... + (-1)^{n-2} a_n \geq 0$ (exercise: deduce from the above explanations that the last inequality holds).
3. $S_{2(n+1)} \geq S_{2n}$ and $S_{2(n+1)+1} = S_{2n+1} - a_{2n+2} + a_{2n+3} \leq S_{2n+1}$. why?

Properties 1, 2, and 3 imply that the sequences (S_{2n}) and (S_{2n+1}) are monotone and bounded and therefore the limits $\underline{\ell} \stackrel{\text{def}}{=} \lim_{n\to\infty} S_{2n}$ and $\overline{\ell} \stackrel{\text{def}}{=} \lim_{n\to\infty} S_{2n+1}$ exist.

Suppose now that $\lim_{n\to\infty} a_n = 0$. Then

$$\overline{\ell} - \underline{\ell} = \lim_{n\to\infty} (S_{2n+1} - S_{2n}) = \lim_{n\to\infty} a_{2n+1} = 0.$$

In other words, $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n}$ and hence the sequence S_n has a limit and moreover $\lim_{n\to\infty} S_n = \overline{\ell} = \underline{\ell}$.

Suppose that $S = \lim_{n\to\infty} S_n$ exists. Then, as we know, each subsequence of this sequence has the same limit. In particular, $S = \lim_{n\to\infty} S_{n+1}$. Hence $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}(S_{n+1} - S_n) = S - S = 0$.
The theorem is proved. □
The last three lines of the proof do not use the the fact that we deal with an alternating series. In fact they prove the following general property of a convergent series:

The Necessary Condition for the Convergence of a Series

Theorem 0.2
Suppose that a series $S = \sum_{n=1}^{\infty} a_n$ *converges. Then* $\lim_{n\to\infty} a_n = 0$.

Proof. See the last three lines of the proof of the previous theorem. □