LONDON TAUGHT COURSE CENTRE

LTCC Basic Course — Statistical Modelling and Estimation

Exercise Sheet 5 — **February/March 2012**

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1. Under the assumptions of and using the notation in Theorem 3.6 show that the test statistic F in the theorem can be written as

$$F = \frac{(\mathbf{Y} - \mathbf{X}_0\hat{\boldsymbol{\gamma}})'(\mathbf{Y} - \mathbf{X}_0\hat{\boldsymbol{\gamma}}) - (\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}})'(\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}})}{\text{MSE}\,\text{rank}(\mathbf{M} - \mathbf{M}_0)},$$

   where $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\gamma}}$ are least squares estimates of the corresponding parameter vectors.

2. In the linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ where $E(\boldsymbol{\epsilon}) = \mathbf{0}$ and $V(\boldsymbol{\epsilon}) = \sigma^2\mathbf{I}$ suppose that the vector $\boldsymbol{\Lambda}'\boldsymbol{\beta}$ is estimable so that $\boldsymbol{\Lambda}' = \mathbf{A}'\mathbf{X}$ for some matrix $\mathbf{A}$. Let $\mathbf{M}$ be the orthogonal projection matrix onto $C(\mathbf{X})$ and $\tilde{\mathbf{M}}$ be the orthogonal projection matrix onto $C(\mathbf{MA})$.

   (a) Show that for every vector $\mathbf{b}$ of appropriate dimension $\mathbf{b}'\mathbf{A}'\mathbf{X} = \mathbf{0}$ if and only if $\mathbf{b}'\mathbf{A}'\mathbf{M} = \mathbf{0}$.

   (b) Hence prove that $\text{rank}(\boldsymbol{\Lambda}) = \text{rank}(\tilde{\mathbf{M}})$.

3. Consider the multiple regression model

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \epsilon_i$$

   for three continuous explanatory variables X_1, X_2, X_3, where it is assumed that $\epsilon_i \sim N(0, \sigma^2)$ and that the random errors ϵ_i and ϵ_j are independent for $i \neq j$.

   (a) For the data in the following table

Y	X_1	X_2	X_3
10	-1	-1	-1.0001
12	-1	1	-0.9999
20	1	-1	1.0000
22	1	1	1.0000
16	0	0	0.0000
18	0	0	0.0000

   test the hypothesis $\beta_1 = \beta_2 = 0$ against a two-sided alternative at the 5% level of significance.

   (b) For the data in (a) also test the hypothesis $\beta_1 = \beta_3 = 0$ against a two-sided alternative, again using a 5% significance level.

4. The two-way analysis of variance model without interaction for two factors A and B with respectively a and b levels where each combination of the levels of A and B is replicated r times is given by

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk},$$

where $i = 1, \ldots, a$, $j = 1, \ldots, b$ and $k = 1, \ldots, r$. Further it is assumed that the random errors ϵ_{ijk} are independent and identically distributed as $N(0, \sigma^2)$.

In what follows suppose that $a = b = 2$ and also that $r = 2$.

(a) Write the model in matrix notation with parameter vector $\boldsymbol{\beta} = [\mu\ \alpha_1\ \alpha_2\ \beta_1\ \beta_2]'$.

(b) Find the test statistic F for testing $H_0 : \boldsymbol{\Lambda}'\boldsymbol{\beta} = \mathbf{0}$, where

$$\boldsymbol{\Lambda}' = \begin{bmatrix} 1 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

and $\boldsymbol{\beta} = [\mu\ \alpha_1\ \alpha_2\ \beta_1\ \beta_2]'$. Also state the distribution of F under H_0.