LONDON TAUGHT COURSE CENTRE

LTCC Basic Course Statistical Modelling and Estimation

Exercise Sheet 4 **February/March 2012**

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1. Consider the normal linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where the components $\epsilon_1, \ldots, \epsilon_n$ of $\boldsymbol{\epsilon}$ are independent random variables with $\epsilon_i \sim N(0, \sigma^2)$ for $i = 1, \ldots, n$. Derive the maximum likelihood estimator of σ^2.

2. Let $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $E(\boldsymbol{\epsilon}) = \mathbf{0}$ and $V(\boldsymbol{\epsilon}) = \sigma^2\mathbf{I}$, be a linear model and suppose that $\sigma^2 > 0$. Prove that the BLUE of any estimable function $\boldsymbol{\lambda}'\boldsymbol{\beta}$ is unique and hence equal to the least squares estimate $\boldsymbol{\lambda}'\hat{\boldsymbol{\beta}}$.

3. A random variable T has a t_m distribution, that is a t distribution with m degrees of freedom, if it can be written as

$$T = \frac{X}{\sqrt{Y/m}}$$

where X and Y are independent random variables which respectively have a standard normal and a chi-square distribution with m degrees of freedom, that is $X \sim N(0, 1)$ and $Y \sim \chi^2_m$. By using this relationship and Theorems 3.1-3.3 from the lectures prove Theorem 3.4 in the lecture notes.

4. Consider the full second-order polynomial regression model (see Example 1.6)

$$Y_i = \beta_0 + \sum_{r=1}^{3} \beta_r x_{ir} + \sum_{r=1}^{3} \beta_{rr} x_{ir}^2 + \sum_{r=1}^{2} \sum_{s=r+1}^{3} \beta_{rs} x_{ir} x_{is} + \epsilon_i,$$

with independent random errors $\epsilon_i \sim N(0, \sigma^2)$ for the half-replicate fractional factorial design for three factors, each at two levels, plus two centre points shown below:

X_1	X_2	X_3
-1	-1	-1
-1	1	1
1	-1	1
1	1	-1
0	0	0
0	0	0

   (a) Show that the function $\beta_1 - \beta_{23}$ of the model parameters is estimable.

   (b) Assuming that the least squares estimate of $\beta_1 - \beta_{23}$ is equal to $\widehat{\beta_1 - \beta_{23}} = 10.03$ and and that the mean square error is MSE $= 5.34$, find a 95% confidence interval for $\beta_1 - \beta_{23}$. How would the interval be interpreted?

5. Suppose that $\mathbf{M}$ and $\mathbf{M}_0$ are orthogonal projection matrices onto $C(\mathbf{M})$ and $C(\mathbf{M}_0)$, where $C(\mathbf{M}_0) \subset C(\mathbf{M})$.

   (a) Prove that $C(\mathbf{M} - \mathbf{M}_0)$ is equal to the orthogonal complement of $C(\mathbf{M}_0)$ with respect to $C(\mathbf{M})$, that is $C(\mathbf{M} - \mathbf{M}_0) = C(\mathbf{M}) \cap C(\mathbf{M}_0)^{\perp}$.

   (b) Show that $\mathbf{M} - \mathbf{M}_0$ is symmetric and idempotent.