LONDON TAUGHT COURSE CENTRE

LTCC Basic Course — Statistical Modelling and Estimation

Exercise Sheet 3 — **February/March 2012**

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1. Consider a $q \times r$ matrix $\mathbf{A}$ and an $r \times q$ matrix $\mathbf{B}$. Show that $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$.

2. Let $\mathbf{Y} = [Y_1, \ldots, Y_n]'$ be a random vector (not necessarily in a linear model) with expectation vector $E(\mathbf{Y}) = [E(Y_1), \ldots, E(Y_n)]'$. The variance-covariance matrix of $\mathbf{Y}$ is defined as
$$V(\mathbf{Y}) = E[(\mathbf{Y} - E(\mathbf{Y}))(\mathbf{Y} - E(\mathbf{Y}))'].$$
Show that if $\mathbf{A}$ is a fixed $m \times n$ matrix, then
$$V(\mathbf{AY}) = \mathbf{A}V(\mathbf{Y})\mathbf{A}'.$$

3. For random vectors $\mathbf{U} = [U_1, \ldots, U_n]'$ with expectation $E(\mathbf{U}) = [E(U_1), \ldots, E(U_n)]'$ and $\mathbf{V} = [V_1, \ldots, V_m]'$ with expectation vector $E(\mathbf{V}) = [E(V_1), \ldots, E(V_m)]'$ the covariance matrix of $\mathbf{U}$ and $\mathbf{V}$ is defined as
$$Cov(\mathbf{U}, \mathbf{V}) = E[(\mathbf{U} - E(\mathbf{U}))(\mathbf{V} - E(\mathbf{V}))'].$$
Show that if $\mathbf{A}$ is a fixed $a \times n$ matrix and $\mathbf{B}$ is a fixed $b \times m$ matrix, then
$$Cov(\mathbf{AU}, \mathbf{BV}) = \mathbf{A}Cov(\mathbf{U}, \mathbf{V})\mathbf{B}'.$$

4. Let $\mathbf{Y} = [Y_1, \ldots, Y_n]'$ be a random vector (not necessarily in a linear model) with $E(\mathbf{Y}) = \boldsymbol{\mu}$ and $V(\mathbf{Y}) = \mathbf{V}$, where $\mathbf{V}$ is a fixed matrix. Prove that if $\mathbf{A}$ is a known $n \times n$ matrix, then
$$E(\mathbf{Y}'\mathbf{AY}) = \mathrm{tr}(\mathbf{AV}) + \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}.$$

5. Consider the one-way analysis of variance model in Example 1.3 for $t = 3$ treatments with design matrix $\mathbf{X}$ and parameter vector $\boldsymbol{\beta} = [\mu \ \alpha_1 \ \alpha_2 \ \alpha_3]'$. Assume that each treatment has replication $r = 3$ and that the data vector is given by
$$\mathbf{y} = [6.48, 5.55, 5.05, 7.07, 6.88, 8.36, 3.84, 4.43, 4.20]'$$
where the first three responses are for treatment 1, the next three responses are for treatment 2 and the final three responses are for treatment 3.

   (a) Write down the design matrix $\mathbf{X}$ and find its rank.

   (b) Is the parameter μ estimable?

   (c) Is the linear function $\alpha_1 - \frac{1}{2}(\alpha_2 + \alpha_3)$ estimable?

   (d) For whichever of the functions in (b) and (c) is estimable find its least squares estimate using the data given and also estimate the variance of the estimator.