LONDON TAUGHT COURSE CENTRE

LTCC Basic Course Statistical Modelling and Estimation

Exercise Sheet 2 **February/March 2012**

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1. Let $\mathbf{X}$ be the design matrix of a linear model. Show that if $\mathbf{G}$ and $\mathbf{H}$ are g-inverses of $\mathbf{X}'\mathbf{X}$, then

   (a) $\mathbf{X}\mathbf{G}\mathbf{X}'\mathbf{X} = \mathbf{X}\mathbf{H}\mathbf{X}'\mathbf{X} = \mathbf{X}$,

   (b) $\mathbf{X}\mathbf{G}\mathbf{X}' = \mathbf{X}\mathbf{H}\mathbf{X}'$.

2. Let $\mathbf{X}$ be the design matrix of a linear model and $\mathbf{M} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'$. Show that regardless of the choice of $(\mathbf{X}'\mathbf{X})^{-}$

   (a) $\mathbf{M}^2 = \mathbf{M}$,

   (b) $\mathbf{M}$ is symmetric.

3. Consider a linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ with $E(\boldsymbol{\epsilon}) = \mathbf{0}$ and $V(\boldsymbol{\epsilon}) = \sigma^2\mathbf{I}$. For $S(\boldsymbol{\beta}) = (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})$ show that

$$\frac{\partial S}{\partial \boldsymbol{\beta}} = -2\mathbf{X}'\mathbf{Y} + 2\mathbf{X}'\mathbf{X}\boldsymbol{\beta}.$$

4. Consider the linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $E(\boldsymbol{\epsilon}) = \mathbf{0}$ and $V(\boldsymbol{\epsilon}) = \sigma^2\mathbf{I}$.

   (a) Prove the equivalence of parts (a) and (c) of Theorem 2.3, that is show that $\boldsymbol{\lambda}'\boldsymbol{\beta}$ is estimable if and only if $\boldsymbol{\lambda}$ is in the column space $C(\mathbf{X}'\mathbf{X})$ of the matrix $\mathbf{X}'\mathbf{X}$.

   (b) Use the equivalence you proved in part (a) to prove Theorem 2.4, that is show that $\boldsymbol{\lambda}'\boldsymbol{\beta}$ is estimable if and only if

$$\boldsymbol{\lambda}'\{\mathbf{I} - (\mathbf{X}'\mathbf{X})^{-}(\mathbf{X}'\mathbf{X})\} = \mathbf{0}.$$

   (c) Show that all linear functions $\boldsymbol{\lambda}'\boldsymbol{\beta}$ are estimable, if and only if $\text{rank}(\mathbf{X}) = p$.

5. Consider the multiple regression model

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i$$

   for $i = 1, \ldots, 4$ with design matrix

$$\mathbf{X} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

   and parameter vector $\boldsymbol{\beta} = [\beta_0\ \beta_1\ \beta_2]'$. The random errors $\epsilon_1, \ldots, \epsilon_4$ are assumed to be uncorrelated with mean zero and constant variance σ^2.

   (a) Find $\text{rank}(\mathbf{X})$.

   (b) Find a generalized inverse of $\mathbf{X}'\mathbf{X}$.

   (c) Hence find a least squares estimator of $\boldsymbol{\beta}$.

   (d) Check whether or not β_1 is estimable.

   (e) Check whether or not $\beta_1 + \beta_2$ is estimable.