Writing Mathematics at Advanced Level: Part I

Franco Vivaldi School of Mathematical Sciences

October 2018

- This session: small-scale features
 - words
 - symbols
 - formulae
 - definitions

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- Following two sessions: techniques for digital presentations.

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BAD: the area of the unit circle GOOD: the area of the unit disc

Exercises

Correct/improve the following expressions:

- 1. The discriminant is < 0.
- 2. 127 is a prime number.
- 3. \sin^2 is positive.
- 4. This function crosses the x-axis twice.
- 5. The solution of $x^2 1 < 0$.
- 6. Consider Θ_n , n < 5.
- 7. The proof splits into 4 cases.
- 8. Add p to q k times.
- 9. The set \mathbb{Q} minus \mathbb{Z} .
- 10. When x > 3, there is no solution.

Correct/improve the following expressions:

- 1. $x^2 + 1$ has no real solution.
- 2. The function g is a function of both x and y.
- 3. We note the fact that S has integer coefficients.
- 4. An example of a trigonometric function is sin.
- 5. We square the equation.
- 6. Purely immaginary is when the real part is zero.
- 7. There are less solutions than for the previous case.
- 8. The solution is not independent of s.
- 9. Thus $x = \alpha$. (We assume that α is positive).
- 10. Remember to always check the sign.

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• Describe with words:
1.
$$\{1, -3, 5, \dots, (-1)^n (2n+1), \dots\}$$

2. $\{(b_1, b_2), (b_2, b_3), \dots, (b_{n-1}, b_n)\}$
3. $\int D(\alpha, \beta) d\alpha$
4. $\frac{\partial H(\theta_1, \dots, \theta_n)}{\partial \theta_1} + \dots + \frac{\partial H(\theta_1, \dots, \theta_n)}{\partial \theta_n}$
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• Provide attributes for the word **equation**.

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$$\alpha = \mathbf{r} + \mathbf{s}\sqrt{2}$$
 $\mathbf{r}, \mathbf{s} \in \mathbb{Q};$ $\mathbf{a}, \mathbf{b} \in \mathbb{Q}, \quad \mathbf{u}, \mathbf{v} \in \mathbb{R} \setminus \mathbb{Q}.$

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Complex: Default is *z*, *w*.

$$z = x + iy$$
 $w = \rho e^{i\theta}$

In analysis, one finds u + iv; number theorists use $\sqrt{-1}$, not i.



—FUNCTIONS:

Default is f, g, h, or Greek letters. Upper-case letters are appropriate for functions of several variables:

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-Sequences:

Vast choice of notation: select the most economical.

$$(a_k)$$
 $(a_k)_{k \ge 0}$ $(a_k)_{k=0}^{n-1}$ (a_1, \ldots, a_n) (a_1, a_2, \ldots)

Use matching symbols:

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$$V_k = (v_1^{(k)}, \dots, v_n^{(k)}), \qquad k = 1, 2, \dots$$
$$V_k = (v_j^{(k)}) \qquad 1 \leq j \leq n, \quad k \geq 1.$$

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Omit parentheses at the exponent if no ambiguity arises; alternatively, use double subscripts

$$V_k = (v_{1,k}, \ldots, v_{n,k}), \qquad k = 1, 2, \ldots$$



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$$\sum_{k=1}^{\infty} a_k \qquad \sum_{k \ge 1} a_k \qquad \sum_k \binom{n}{k} k.$$

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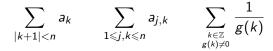
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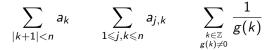
Example: defining Euler φ -function with a sum

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The operators \prod, \bigcap, \bigcup have the same syntax as \sum :

$$n! = \prod_{k=1}^{n} k \qquad \bigcup_{n \in \mathbb{N} \atop n \text{ prime}} \mathbb{Z}[n^{-1}]$$

Improve the notation.

1.
$$a = (a_1, a_2, a_3, ..., a_t)$$
.
2. $f(x) = \frac{14x - 2x^3 - 2x^2 + 14}{-2x - 4}$
3. $\left\{ k \in \mathbb{Q} : k = \frac{x}{x^2 + 1}, x \in \mathbb{Z}, x < 0 \right\}$
4. Let β_{α} be a one-parameter family of vectors in C.
5. Let A, B be sets, and let $p \in A, r \in B$.
6. $\mu : A \to B, \quad \mu(\lambda) = \sin(\lambda \pi)$
7. $h(x) = f \circ g(x)$
8. $\sum_{k=1}^{n+1} a_{k+1}$

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Writing well

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$$z(x, \mathbf{n}) = \sum_{k=1}^{\infty} k^2 \sum_{n=0}^{n_k-1} f(x+n), \quad \mathbf{n} = (n_1, n_2 \dots)$$

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Let $\mathbf{A}_k = \{A_1, \dots, A_k\}$, and let $\mathbb{A} = \bigcap_{k \ge 1} \mathbf{A}_k$.

-Use quantifiers sparingly:

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GOOD: Hence, if x > 0, then $x \in B$.

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These expressions are concise and elegant —if cryptic.

Exercises

Describe with words

1. Q^3 N^ℤ 3. $| \mathbb{N}^n$ *n*≥1 4. $a\mathbb{Z} + b\mathbb{Z}$ 5. $\Gamma(z+1) = z\Gamma(z)$. 6. $X = \{1, X\}$ 7. $\frac{1}{2}\mathbb{Z}$ 8. $\mathbb{Z}\left[\frac{1}{2}\right]$

Translate the following sentences into words (f is a real function).

1. $4\mathbb{Z} \subset 2\mathbb{Z}$ 2. $\#f(\mathbb{R}) = 1$ 3. $\#f^{-1}(\{0\}) < \infty$ 4. $\forall \gamma, \delta \in \Omega, \gamma \delta = \delta \gamma$ 5. $\forall r \in \mathbb{Q} \setminus \{0\}$: $f(r) \neq 0$ 6. $\forall \alpha \in A, \forall \epsilon > 0, \exists \beta \in B, |\alpha - \beta| < \epsilon$ 7. $\exists a \in \mathbb{R}, \forall x \in \mathbb{R}^+, f(x+a) = f(a)$ 8. $\#\{f(x) : x \in \mathbb{Z}\} = \infty$ 9. $f(\mathbb{R}) = f(\mathbb{Z})$

Turn words into symbols: (f is a real function, unless specified otherwise)

- 1. The function f assumes only integer values.
- 2. The function f is not always positive.
- 3. The function $f : A \rightarrow B$ is not constant.
- 4. The polynomial p(x) has no rational roots.
- 5. The function f vanishes for all sufficiently large arguments.
- 6. There are zeros of f arbitrarily close to the origin of the Cartesian plane.

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$$(1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots)$$

belongs to \mathcal{N} .

The absolute value is a function that associates to every integer x a number |x| with the following properties

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(The definition of *p*-adic absolute values then follows.)

Exercise: re-write for a beginning undergraduate

Let

$$\ln(x) = \int_1^x \frac{1}{t} dt \qquad x > 0.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) = \frac{1}{x} \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}x}\ln(u(x)) = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

for any differentiable u(x) > 0. Then

$$u(x) := ax \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x} \ln(ax) = \frac{1}{ax} \frac{\mathrm{d}}{\mathrm{d}x} (ax) = \frac{1}{ax} \cdot a = \frac{1}{x}.$$

$$\Rightarrow \ln(x)' = \ln(ax)', \text{ so that } \ln(ax) = \ln(x) + C, \text{ and}$$

$$x = 1 \quad \Rightarrow \quad \ln(a \cdot 1) = \ln(a) = \ln(1) + C = 0 + C$$

$$\Rightarrow C = \ln(a), \text{ giving } \ln(ax) = \ln(a) + \ln(x).$$

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Being differentiable, this function is also continuous. The chain rule of differentiation extends to equation (1) to give

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(u(x)) = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

for any positive differentiable function u(x).

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$$\ln(ax) = \ln(x) + C \tag{2}$$

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giving $C = \ln(a)$. Substituting this in equation (2) gives

$$\ln(ax) = \ln(a) + \ln(x) \tag{3}$$

which is the basic property of the logarithmic function.