Writing Mathematics at Advanced Level: Part I

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  - words
  - symbols
  - formulae
  - definitions

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- Following two sessions: techniques for digital presentations.

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BAD: the area of the unit circle
GOOD: the area of the unit disc

Exercises

Correct/improve the following expressions:

1. *The discriminant is* < 0.
2. 127 *is a prime number.*
3. $\sin^2$ *is positive.*
4. *This function crosses the* x*-axis twice.*
5. *The solution of* $x^2 - 1 < 0$.
6. *Consider* Θ_n, $n < 5$.
7. *The proof splits into 4 cases.*
8. *Add* p *to* q k *times.*
9. *The set* $\mathbb{Q}$ *minus* $\mathbb{Z}$.
10. *When* $x > 3$, *there is no solution.*

Correct/improve the following expressions:

1. $x^2 + 1$ *has no real solution.*
2. *The function g is a function of both x and y.*
3. *We note the fact that S has integer coefficients.*
4. *An example of a trigonometric function is* $\sin$*.*
5. *We square the equation.*
6. *Purely immaginary is when the real part is zero.*
7. *There are less solutions than for the previous case.*
8. *The solution is not independent of s.*
9. *Thus $x = \alpha$. (We assume that α is positive).*
10. *Remember to always check the sign.*

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the last **component** (or **entry**) of the vector $\mathbf{v}$

Exercises

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- Describe with words:
  1. $\{1, -3, 5, \ldots, (-1)^n(2n+1), \ldots\}$
  2. $\{(b_1, b_2), (b_2, b_3), \ldots, (b_{n-1}, b_n)\}$
  3. $\int D(\alpha, \beta)\mathrm{d}\alpha$
  4. $\dfrac{\partial H(\theta_1, \ldots, \theta_n)}{\partial \theta_1} + \cdots + \dfrac{\partial H(\theta_1, \ldots, \theta_n)}{\partial \theta_n}$
  5. $A_1 \supset A_2 \supset A_3 \supset \cdots$
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- Provide attributes for the word **equation**.

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$$\alpha = r + s\sqrt{2} \quad r, s \in \mathbb{Q}; \qquad a, b \in \mathbb{Q}, \quad u, v \in \mathbb{R} \setminus \mathbb{Q}.$$

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Complex: Default is z, w.

$$z = x + \mathrm{i}y \qquad w = \rho e^{\mathrm{i}\theta}$$

In analysis, one finds $u + \mathrm{i}v$; number theorists use $\sqrt{-1}$, not i.

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Default is f, g, h, or Greek letters. Upper-case letters are appropriate for functions of several variables:

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Vast choice of notation: select the most economical.

$$(a_k) \qquad (a_k)_{k\geqslant 0} \qquad (a_k)_{k=0}^{n-1} \qquad (a_1, \ldots, a_n) \qquad (a_1, a_2, \ldots)$$

Use matching symbols:

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$$V_k = (v_1^{(k)}, \ldots, v_n^{(k)}), \qquad k = 1, 2, \ldots$$

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Omit parentheses at the exponent if no ambiguity arises; alternatively, use double subscripts

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$$\sum_{|k+1|<n} a_k \qquad \sum_{1\leqslant j,k\leqslant n} a_{j,k} \qquad \sum_{\substack{k\in\mathbb{Z}\\ g(k)\neq 0}} \frac{1}{g(k)}$$

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The operators $\prod, \bigcap, \bigcup$ have the same syntax as $\sum$:

$$n! = \prod_{k=1}^{n} k \qquad \bigcup_{\substack{n\in\mathbb{N}\\ n \text{ prime}}} \mathbb{Z}[n^{-1}]$$

Exercises

Improve the notation.

1. $a = (a_1, a_2, a_3, \ldots, a_t)$.
2. $f(x) = \dfrac{14x - 2x^3 - 2x^2 + 14}{-2x - 4}$
3. $\left\{k \in \mathbb{Q} : k = \dfrac{x}{x^2+1}, x \in \mathbb{Z}, x < 0\right\}$
4. *Let β_α be a one-parameter family of vectors in C.*
5. *Let A, B be sets, and let $p \in A, r \in B$.*
6. $\mu : A \to B, \qquad \mu(\lambda) = \sin(\lambda\pi)$
7. $h(x) = f \circ g(x)$
8. $\displaystyle\sum_{k=1}^{n+1} a_{k+1}$

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GOOD: $$z(x, \mathbf{n}) = \sum_{k=1}^{\infty} k^2 \sum_{n=0}^{n_k - 1} f(x + n), \qquad \mathbf{n} = (n_1, n_2 \dots)$$

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> *For all real functions f, if f is differentiable, then f is continuous.*

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> *Let $\mathbf{A}_k = \{A_1, \ldots, A_k\}$, and let $\mathbb{A} = \bigcap_{k\geqslant 1} \mathbf{A}_k$.*

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These expressions are concise and elegant —if cryptic.

Exercises

Describe with words

1. $\mathbb{Q}^3$
2. $\mathbb{N}^{\mathbb{Z}}$
3. $\bigcup_{n \geqslant 1} \mathbb{N}^n$
4. $a\mathbb{Z} + b\mathbb{Z}$
5. $\Gamma(z+1) = z\Gamma(z)$.
6. $X = \{1, X\}$
7. $\frac{1}{2}\mathbb{Z}$
8. $\mathbb{Z}\left[\frac{1}{2}\right]$

Translate the following sentences into words (f is a real function).

1. $4\mathbb{Z} \subset 2\mathbb{Z}$
2. $\# f(\mathbb{R}) = 1$
3. $\# f^{-1}(\{0\}) < \infty$
4. $\forall \gamma, \delta \in \Omega, \gamma\delta = \delta\gamma$
5. $\forall r \in \mathbb{Q} \setminus \{0\} : f(r) \neq 0$
6. $\forall \alpha \in A, \forall \epsilon > 0, \exists \beta \in B, |\alpha - \beta| < \epsilon$
7. $\exists a \in \mathbb{R}, \forall x \in \mathbb{R}^+, f(x + a) = f(a)$
8. $\#\{f(x) : x \in \mathbb{Z}\} = \infty$
9. $f(\mathbb{R}) = f(\mathbb{Z})$

Turn words into symbols: (f is a real function, unless specified otherwise)

1. *The function f assumes only integer values.*
2. *The function f is not always positive.*
3. *The function $f : A \to B$ is not constant.*
4. *The polynomial $p(x)$ has no rational roots.*
5. *The function f vanishes for all sufficiently large arguments.*
6. *There are zeros of f arbitrarily close to the origin of the Cartesian plane.*

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Let $\mathcal{N}$ be the set of sequences of natural numbers, such that every natural number is listed infinitely often. **?** *For example, the sequence*

$$(1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots)$$

belongs to $\mathcal{N}$.

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The absolute value is a function that associates to every integer x a number $|x|$ with the following properties

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(The definition of p-**adic** absolute values then follows.)

Exercise: re-write for a beginning undergraduate

Let

$$\ln(x) = \int_1^x \frac{1}{t}\mathrm{d}t \qquad x > 0.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) = \frac{1}{x} \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}x}\ln(u(x)) = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

for any differentiable $u(x) > 0$. Then

$$u(x) := ax \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\ln(ax) = \frac{1}{ax}\frac{\mathrm{d}}{\mathrm{d}x}(ax) = \frac{1}{ax}\cdot a = \frac{1}{x}.$$

$\Rightarrow$ $\ln(x)' = \ln(ax)'$, so that $\ln(ax) = \ln(x) + C$, and

$$x = 1 \quad \Rightarrow \quad \ln(a\cdot 1) = \ln(a) = \ln(1) + C = 0 + C$$

$\Rightarrow$ $C = \ln(a)$, giving $\ln(ax) = \ln(a) + \ln(x)$.

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Being differentiable, this function is also continuous. The chain rule of differentiation extends to equation (1) to give

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(u(x)) = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

for any positive differentiable function $u(x)$.

Consider now the function $u(x) = ax$, for some positive real number a. Applying the chain rule we get

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giving $C = \ln(a)$. Substituting this in equation (2) gives

$$\ln(ax) = \ln(a) + \ln(x) \tag{3}$$

which is the basic property of the logarithmic function.