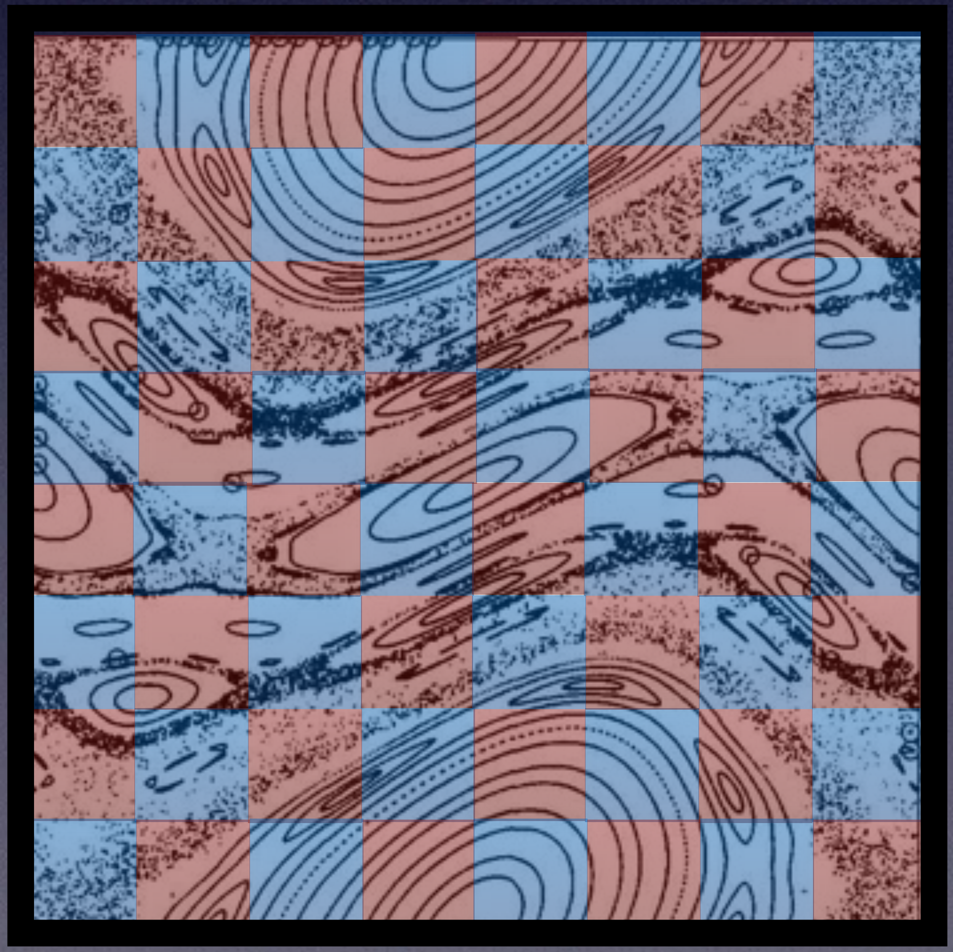
Sample slides

Franco Vivaldi

Hamiltonian stability

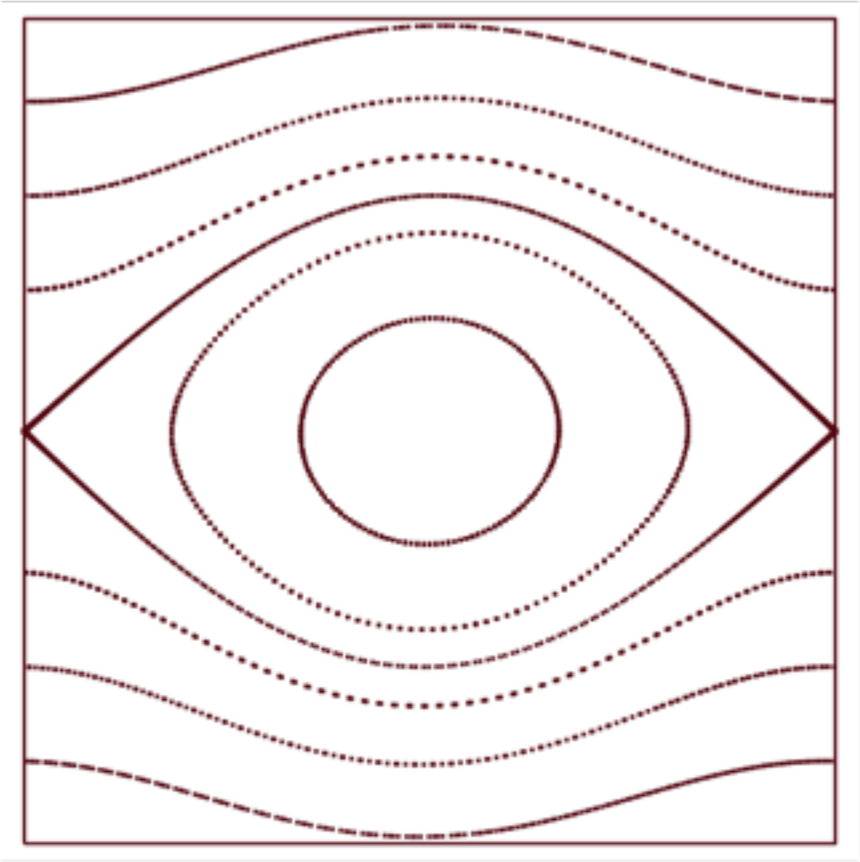
over discrete spaces

Franco Vivaldi
Queen Mary, University of London



Smooth area-preserving maps

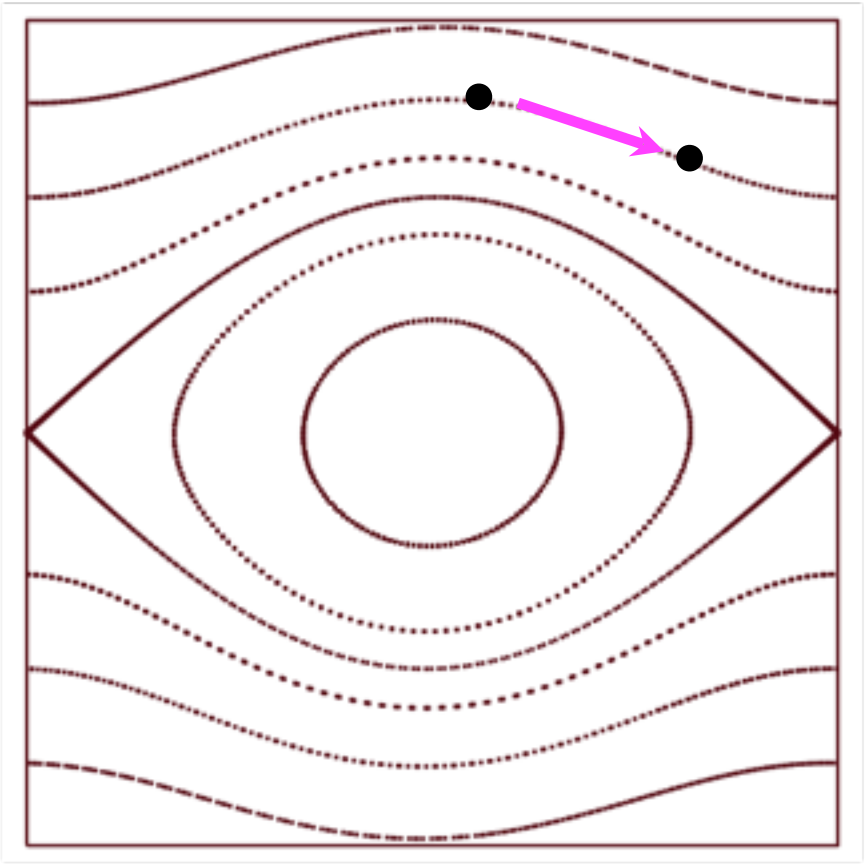
Smooth area-preserving maps



integrable

Smooth area-preserving maps

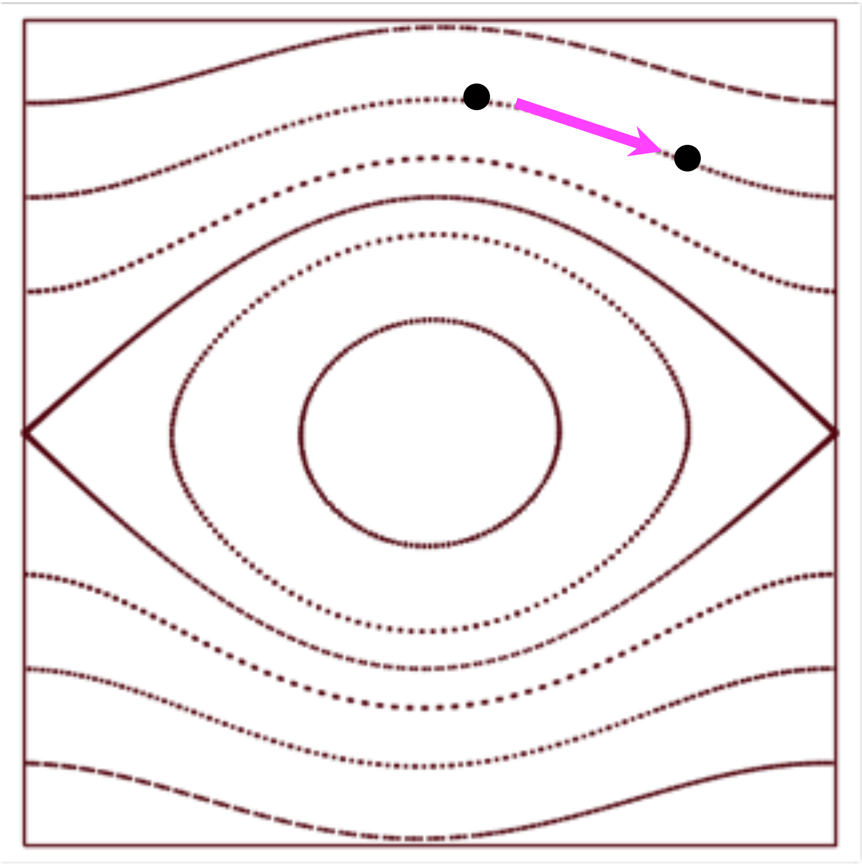
foliation by
invariant curves



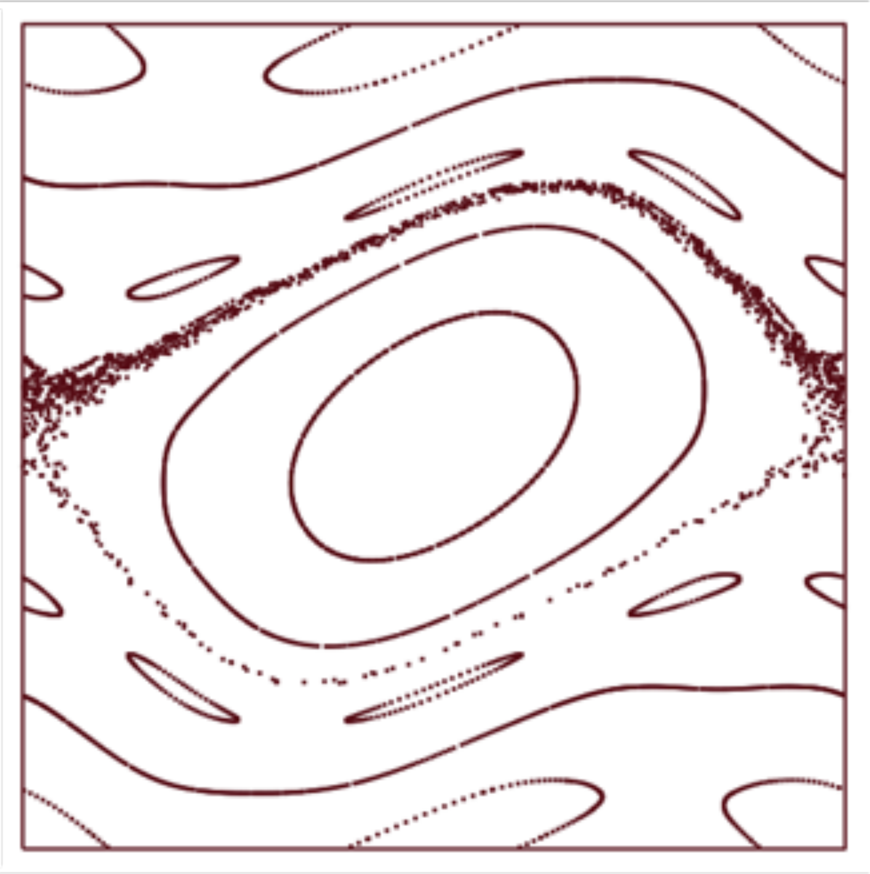
integrable

Smooth area-preserving maps

foliation by
invariant curves



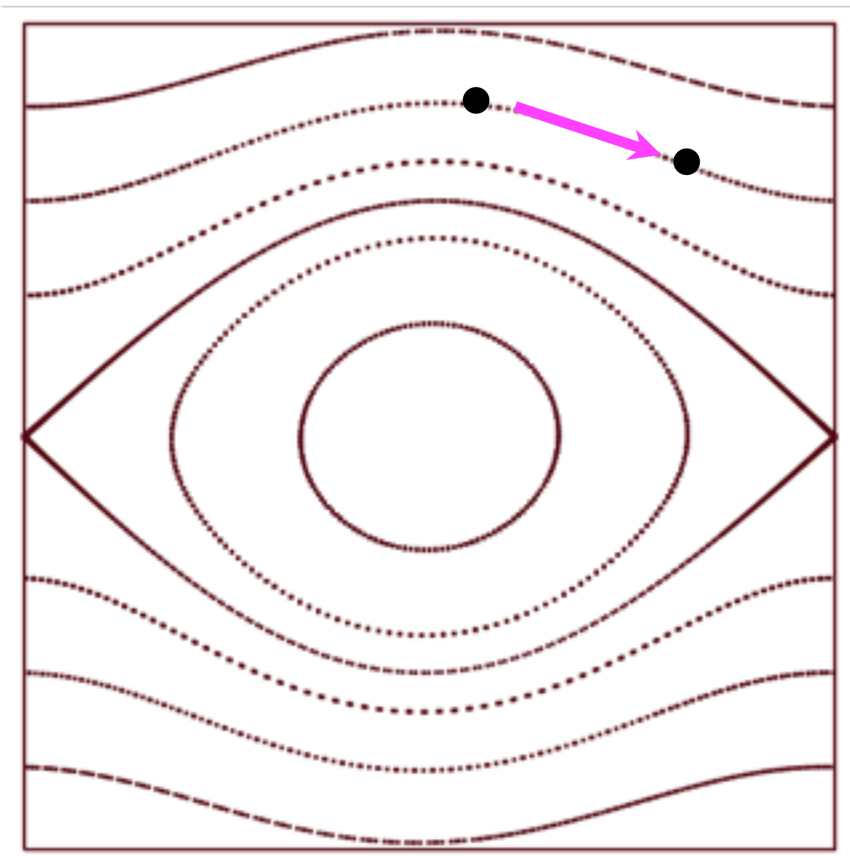
integrable



near-integrable: stable

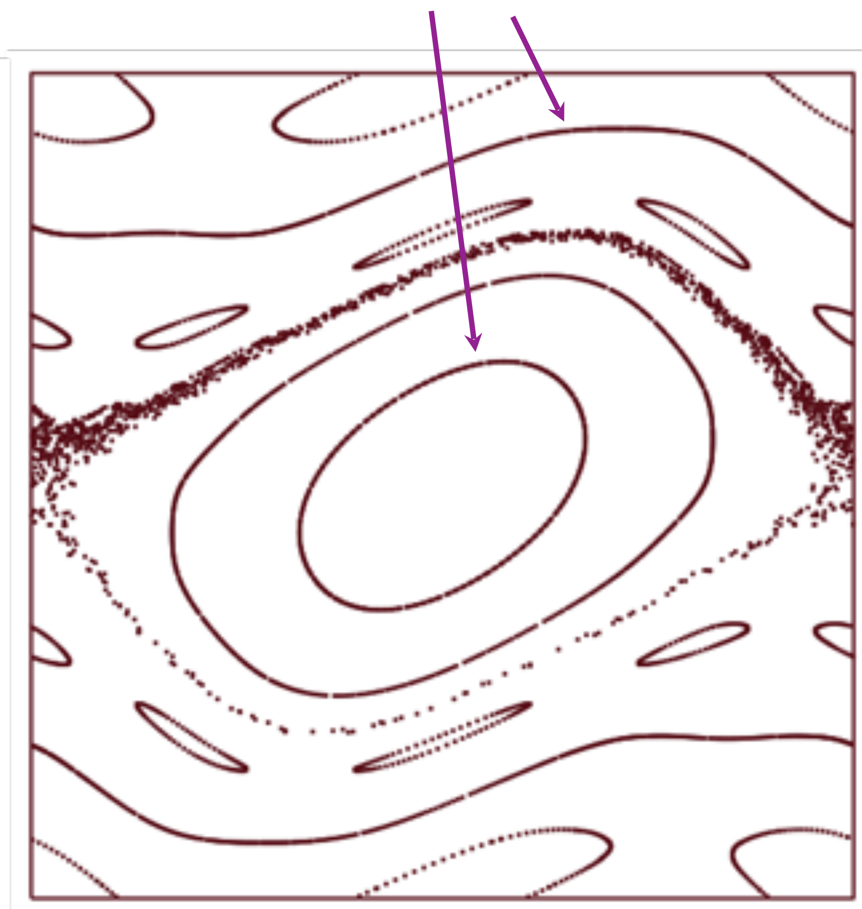
Smooth area-preserving maps

foliation by
invariant curves



integrable

KAM curves

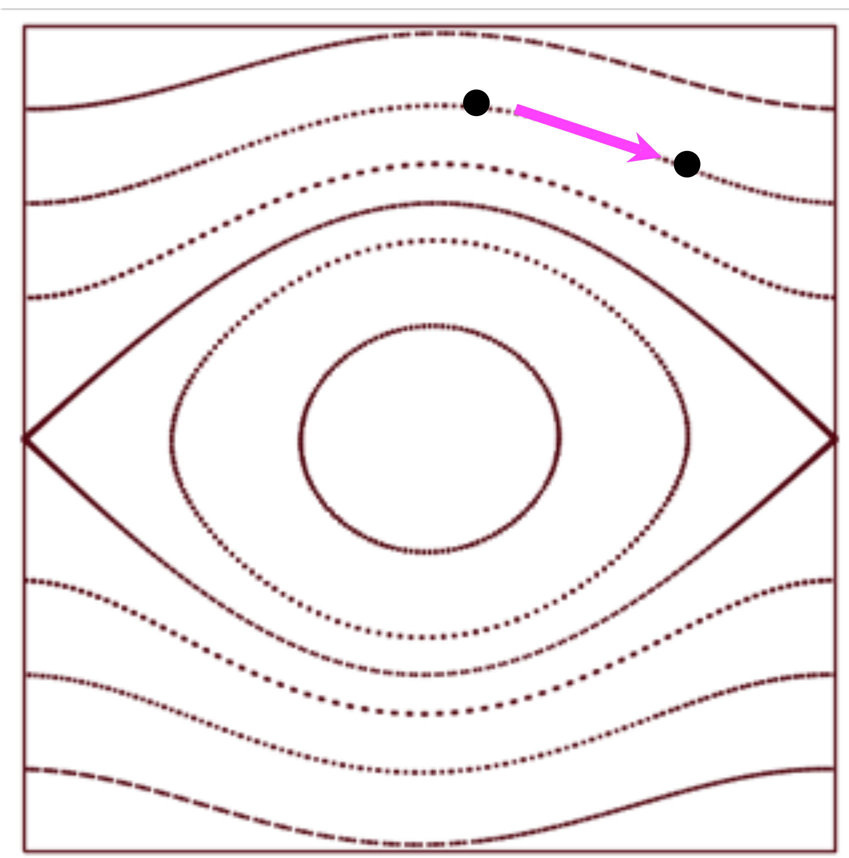


near-integrable: stable

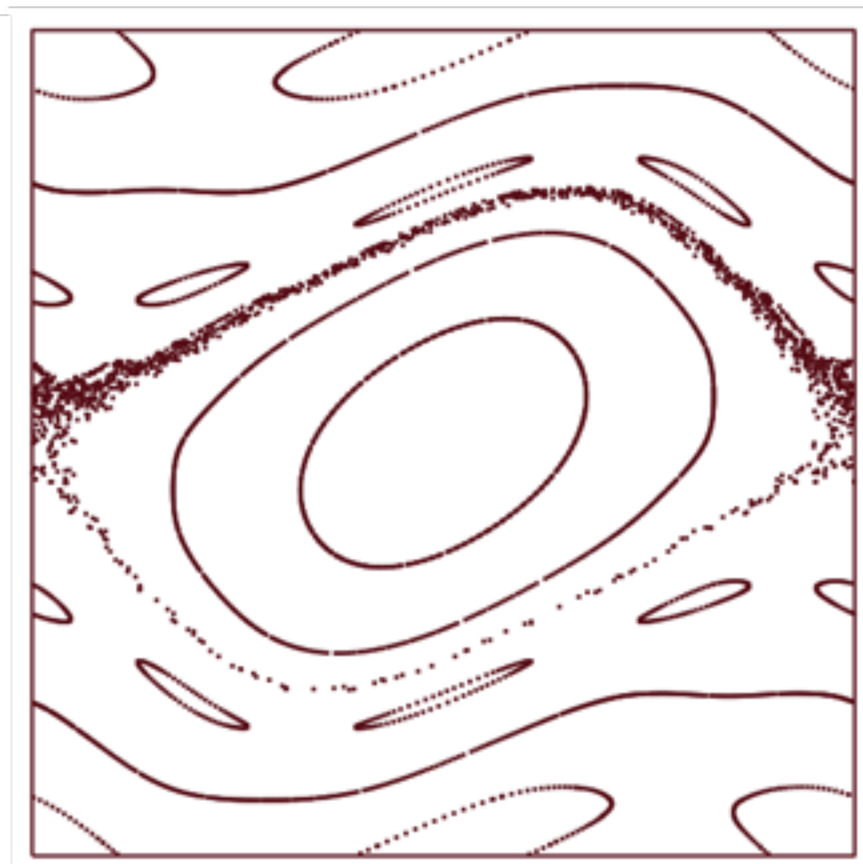
Smooth area-preserving maps

foliation by
invariant curves

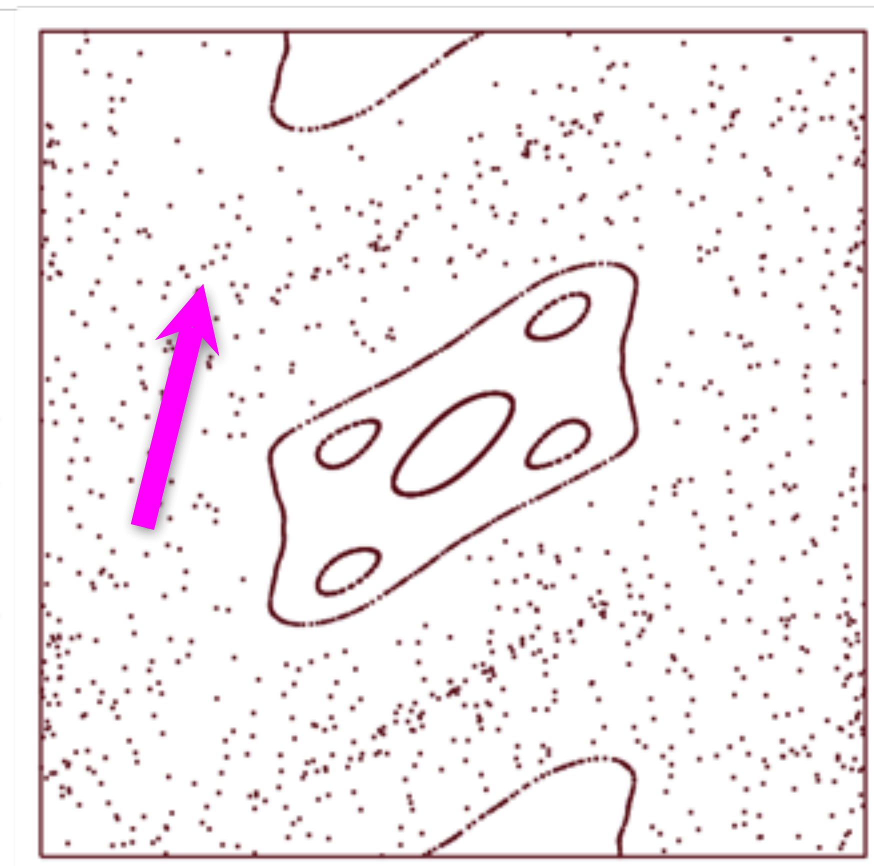
KAM curves



integrable



near-integrable: stable

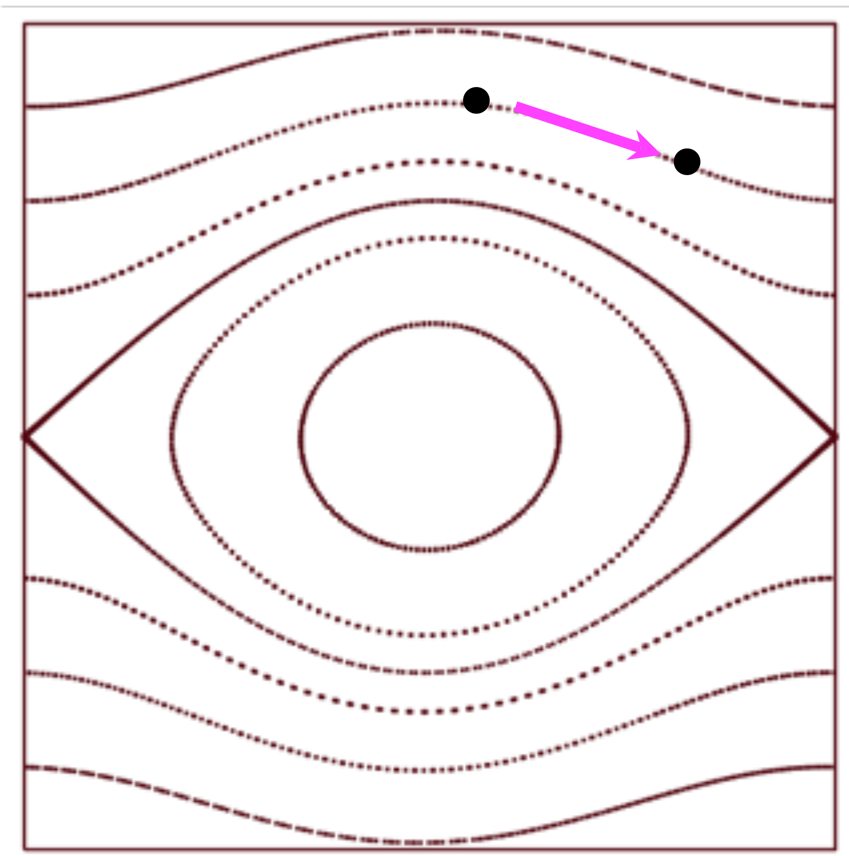


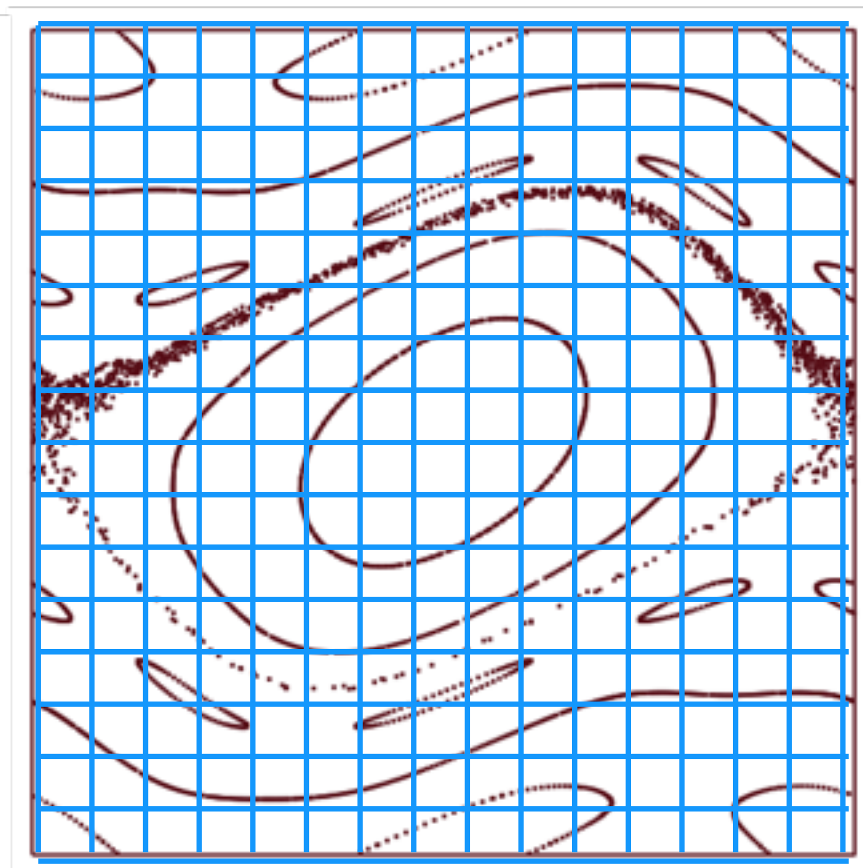
strong perturbation:
unstable

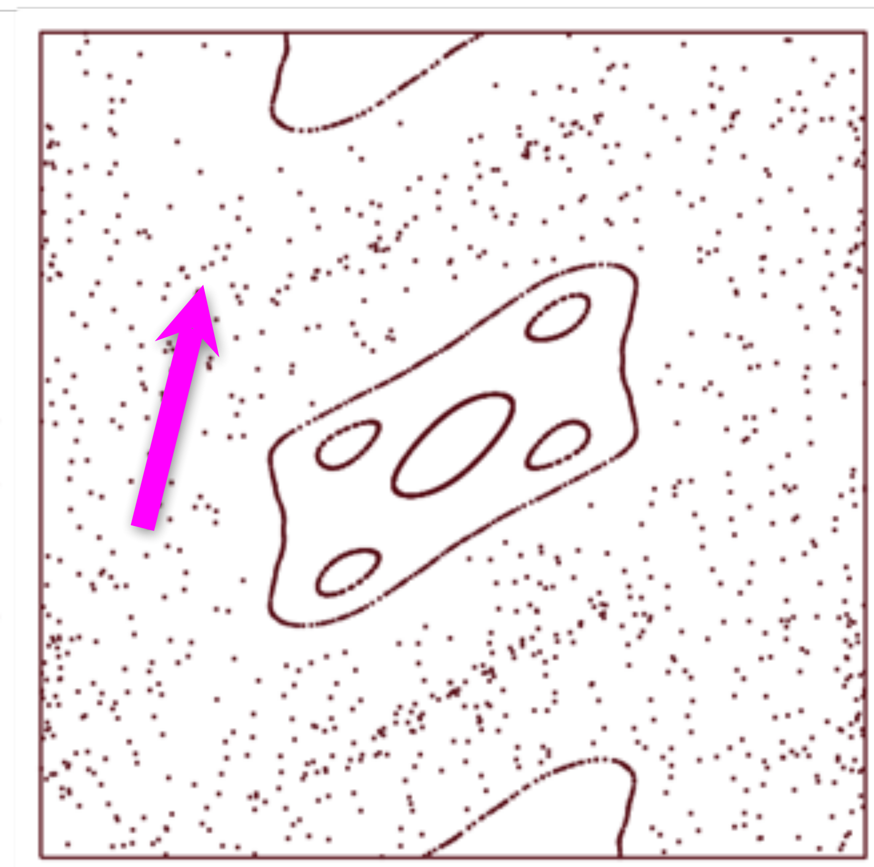
Smooth area-preserving maps

foliation by
invariant curves

KAM curves







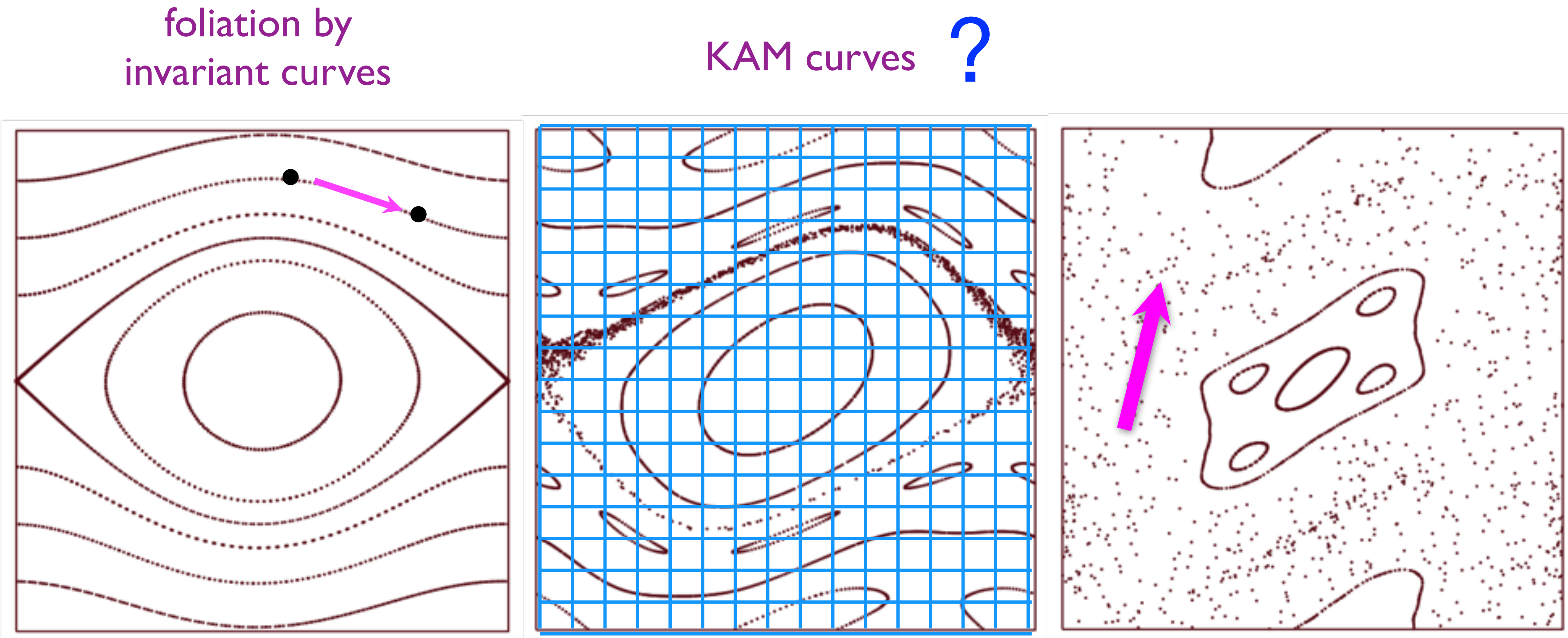
integrable

near-integrable: stable

strong perturbation:
unstable

What happens if the space is discrete?

Smooth area-preserving maps



integrable

near-integrable: stable

strong perturbation: unstable

What happens if the space is discrete?

Some methods for discretizing space

Some methods for discretizing space

- Truncation (computer arithmetic).

Some methods for discretizing space

- Truncation (computer arithmetic).
- Restricting coordinates to a discrete field (ring, module).

Some methods for discretizing space

- Truncation (computer arithmetic).
- Restricting coordinates to a discrete field (ring, module).
- Geometric discretization.

Some methods for discretizing space

- Truncation (computer arithmetic).
- Restricting coordinates to a discrete field (ring, module).
- Geometric discretization.
- Reduction to a finite field.

Early investigations:

F. Rannou (1974)

Early investigations:

F. Rannou (1974)

Astron. & Astrophys. 31, 289 – 301 (1974)

Numerical Study of Discrete Plane Area-preserving Mappings

F. Rannou
Observatoire de Nice

Received August, 10, 1973

Early investigations:

F. Rannou (1974)

Astron. & Astrophys. 31, 289 – 301 (1974)

Numerical Study of Discrete Plane Area-preserving Mappings

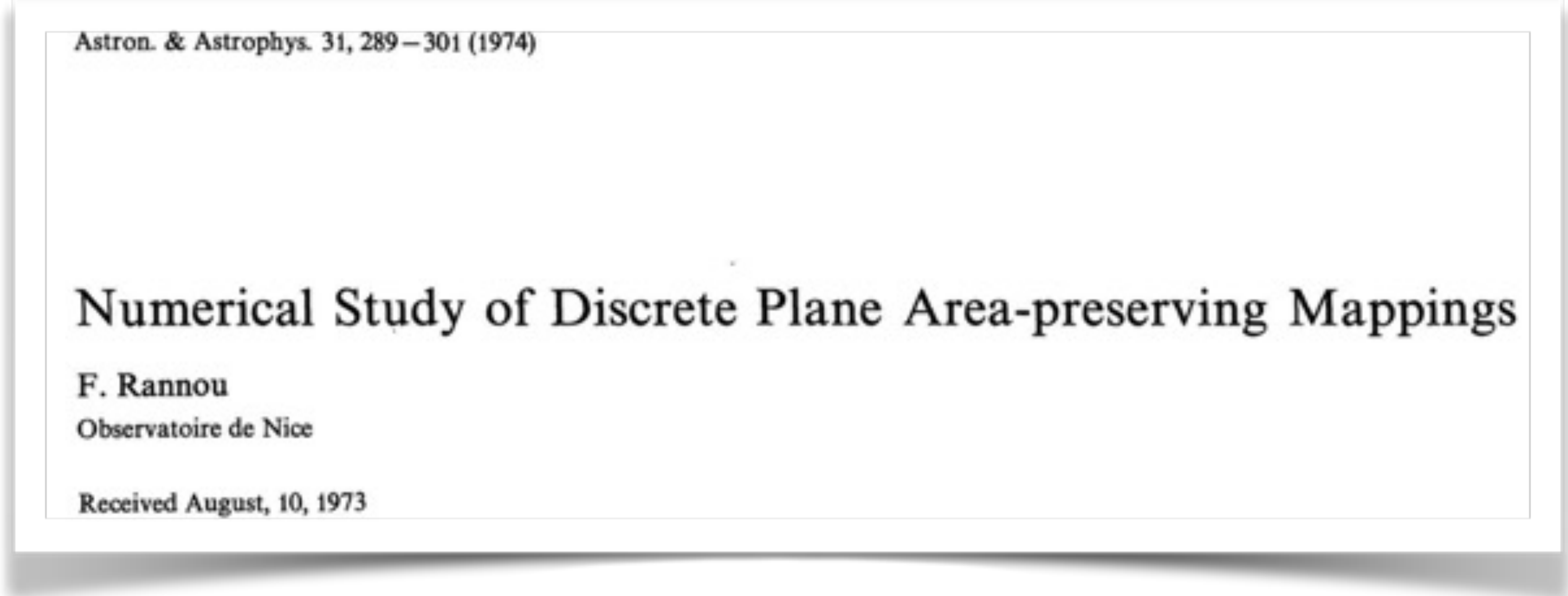
F. Rannou
Observatoire de Nice

Received August, 10, 1973

First study of an invertible lattice map on the torus

Early investigations:

F. Rannou (1974)


Astron. & Astrophys. 31, 289–301 (1974)

Numerical Study of Discrete Plane Area-preserving Mappings

F. Rannou
Observatoire de Nice

Received August, 10, 1973

First study of an invertible lattice map on the torus

$$
\begin{aligned}
x_{t+1} &= x_t + y_t + \frac{m}{2\pi}\left(1 - \cos\frac{2\pi}{m}y_t\right) && (\mathrm{mod}\, m) \\
y_{t+1} &= y_t - \frac{\lambda m}{2\pi}\left(\sin\frac{2\pi}{m}x_{t+1} + 1 - \cos\frac{2\pi}{m}x_{t+1}\right) && (\mathrm{mod}\, m)
\end{aligned}
$$

Early investigations:

F. Rannou (1974)

Astron. & Astrophys. 31, 289–301 (1974)

Numerical Study of Discrete Plane Area-preserving Mappings

F. Rannou
Observatoire de Nice

Received August, 10, 1973

First study of an invertible lattice map on the torus

lattice size

$$x_{t+1} = x_t + y_t + \frac{m}{2\pi}\left(1 - \cos\frac{2\pi}{m}y_t\right) \pmod{m}$$

$$y_{t+1} = y_t - \frac{\lambda m}{2\pi}\left(\sin\frac{2\pi}{m}x_{t+1} + 1 - \cos\frac{2\pi}{m}x_{t+1}\right) \pmod{m}$$

Early investigations:

F. Rannou (1974)

Astron. & Astrophys. 31, 289–301 (1974)

Numerical Study of Discrete Plane Area-preserving Mappings

F. Rannou
Observatoire de Nice

Received August, 10, 1973

First study of an invertible lattice map on the torus

lattice size

$$
\begin{aligned}
x_{t+1} &= x_t + y_t + \left[\frac{m}{2\pi}\left(1-\cos\frac{2\pi}{m}y_t\right)\right] && \pmod{m} \\
y_{t+1} &= y_t - \left[\frac{\lambda m}{2\pi}\left(\sin\frac{2\pi}{m}x_{t+1} + 1 - \cos\frac{2\pi}{m}x_{t+1}\right)\right] && \pmod{m}
\end{aligned}
$$

rounding to nearest integer

Early investigations:

F. Rannou (1974)

Astron. & Astrophys. 31, 289 – 301 (1974)

Numerical Study of Discrete Plane Area-preserving Mappings

F. Rannou
Observatoire de Nice

Received August, 10, 1973

First study of an invertible lattice map on the torus

lattice size

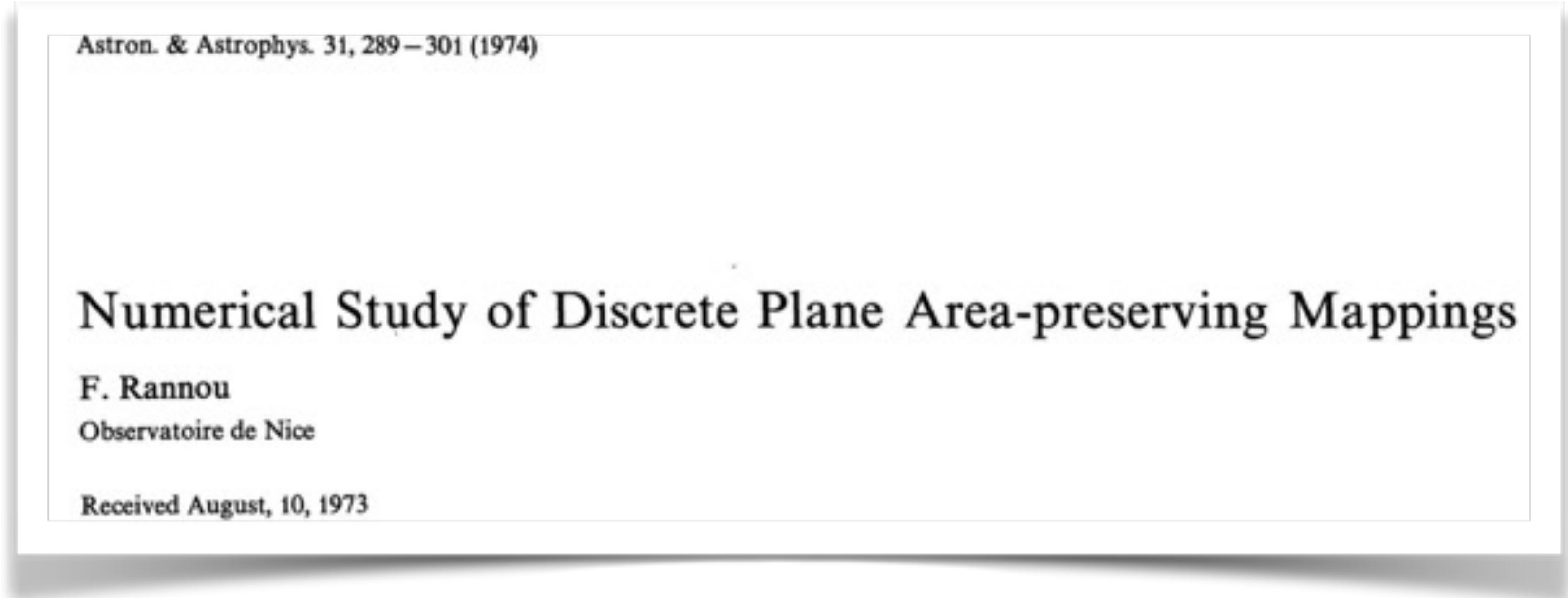
$$
\begin{aligned}
x_{t+1} &= x_t + y_t + \left[\frac{m}{2\pi}\left(1-\cos\frac{2\pi}{m}y_t\right)\right] \pmod{m} \\
y_{t+1} &= y_t - \left[\frac{\lambda m}{2\pi}\left(\sin\frac{2\pi}{m}x_{t+1} + 1 - \cos\frac{2\pi}{m}x_{t+1}\right)\right] \pmod{m}
\end{aligned}
$$

rounding to
nearest integer

- All orbits are periodic.

Early investigations:

F. Rannou (1974)


Astron. & Astrophys. 31, 289–301 (1974)

Numerical Study of Discrete Plane Area-preserving Mappings

F. Rannou
Observatoire de Nice

Received August, 10, 1973

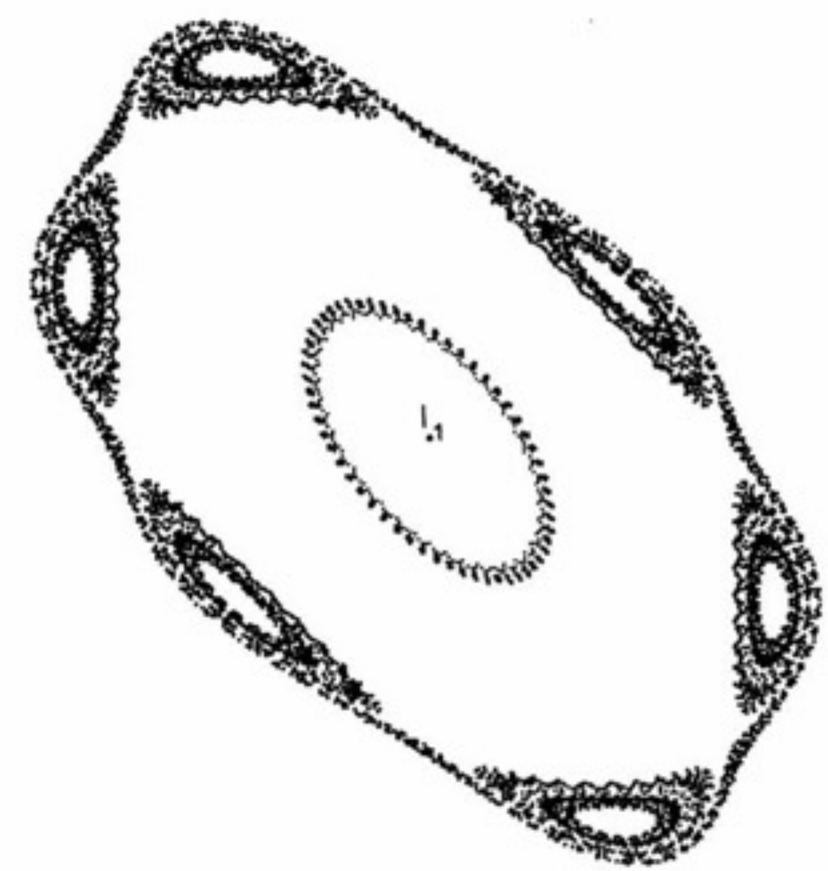
First study of an invertible lattice map on the torus

lattice size

$$
\begin{aligned}
x_{t+1} &= x_t + y_t + \left[\frac{m}{2\pi}\left(1-\cos\frac{2\pi}{m}y_t\right)\right] \pmod{m} \\
y_{t+1} &= y_t - \left[\frac{\lambda m}{2\pi}\left(\sin\frac{2\pi}{m}x_{t+1} + 1 - \cos\frac{2\pi}{m}x_{t+1}\right)\right] \pmod{m}
\end{aligned}
$$

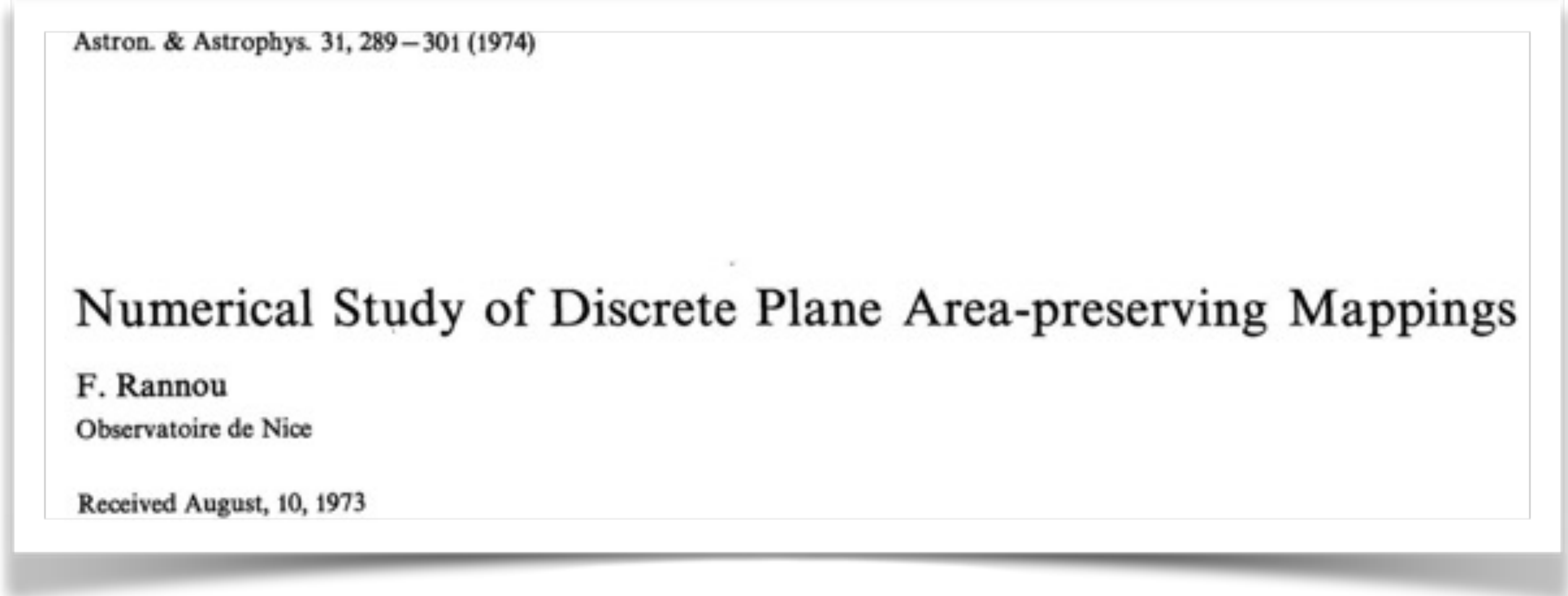
rounding to nearest integer

- All orbits are periodic.
- Orbit representing curves develop some “thickness”, but remain stable.



Early investigations:

F. Rannou (1974)


Astron. & Astrophys. 31, 289 – 301 (1974)

Numerical Study of Discrete Plane Area-preserving Mappings

F. Rannou
Observatoire de Nice

Received August, 10, 1973

First study of an invertible lattice map on the torus

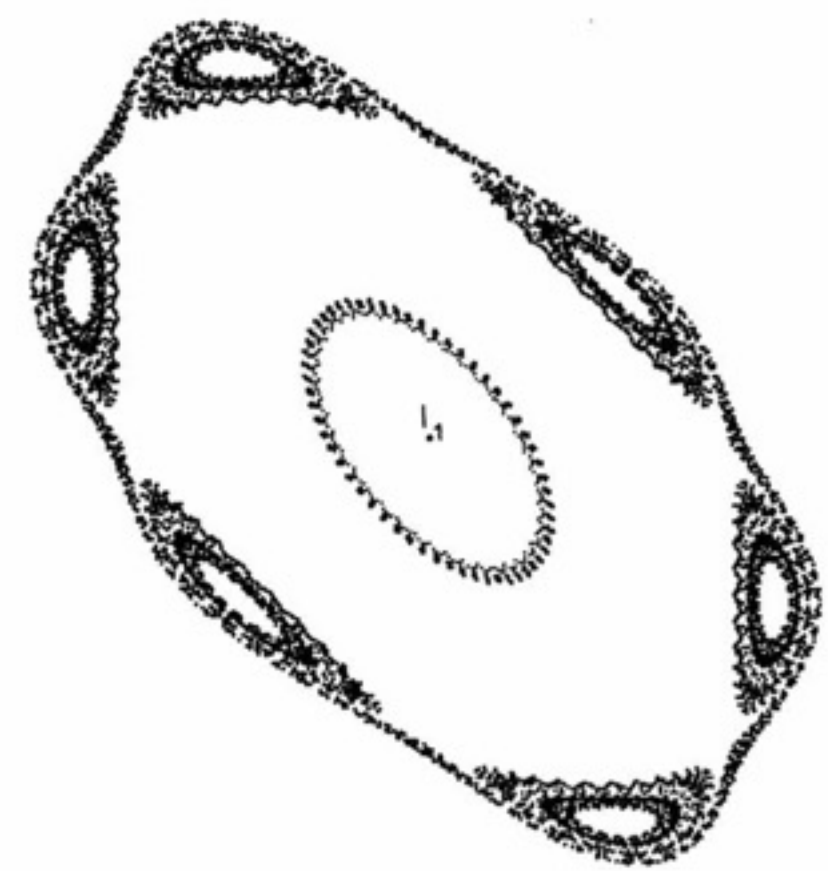
lattice size

$$x_{t+1} = x_t + y_t + \left[\frac{m}{2\pi}\left(1 - \cos\frac{2\pi}{m} y_t\right)\right] \pmod{m}$$

$$y_{t+1} = y_t - \left[\frac{\lambda m}{2\pi}\left(\sin\frac{2\pi}{m} x_{t+1} + 1 - \cos\frac{2\pi}{m} x_{t+1}\right)\right] \pmod{m}$$

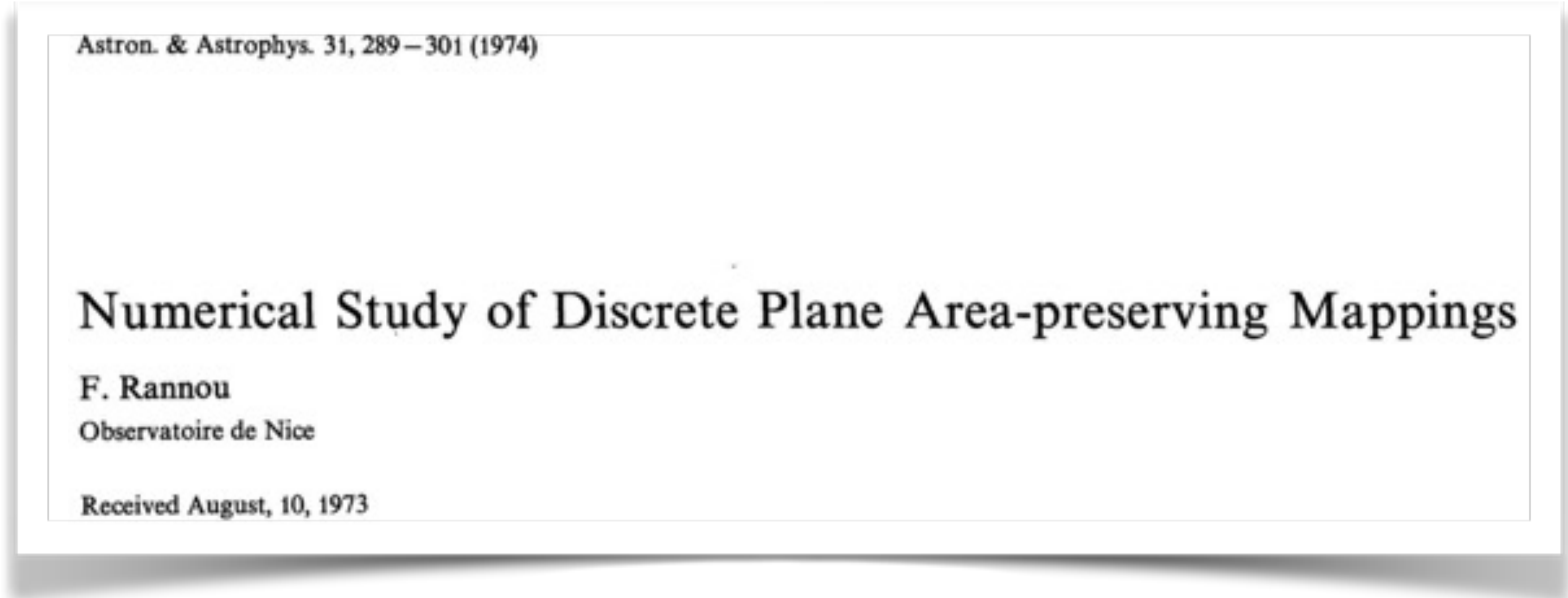
rounding to nearest integer

- All orbits are periodic.
- Orbit representing curves develop some "thickness", but remain stable.
- In the chaotic regions the map behaves like a random permutation.



Early investigations:

F. Rannou (1974)



Astron. & Astrophys. 31, 289 – 301 (1974)

Numerical Study of Discrete Plane Area-preserving Mappings

F. Rannou
Observatoire de Nice

Received August, 10, 1973

First study of an invertible lattice map on the torus

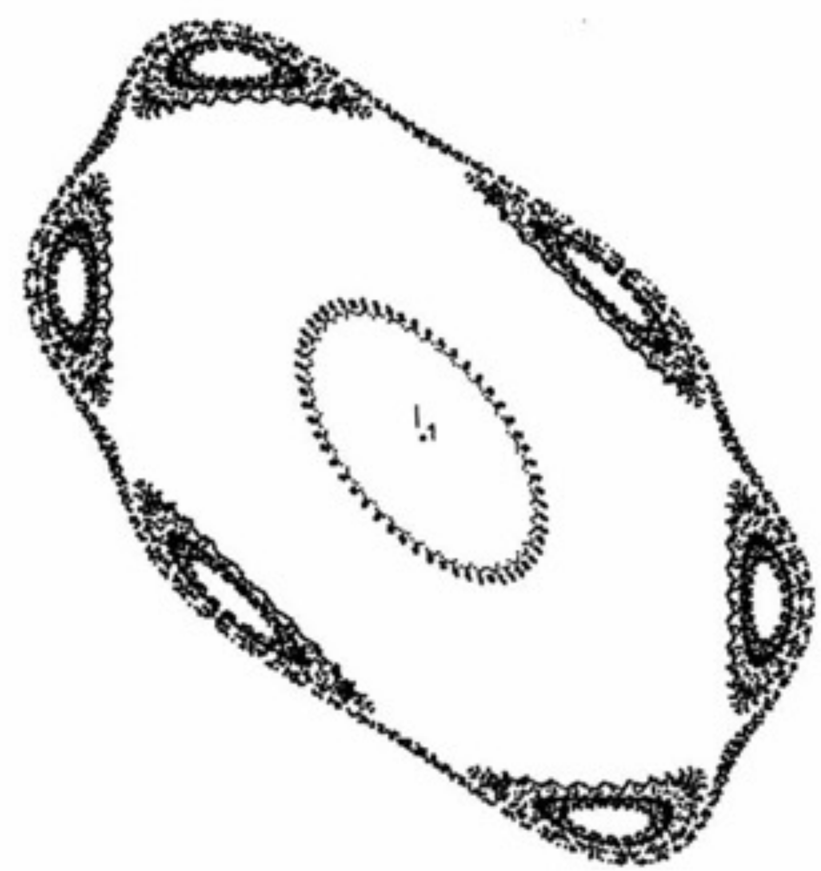
$$x_{t+1} = x_t + y_t + \left[\frac{m}{2\pi}\left(1 - \cos\frac{2\pi}{m}y_t\right)\right] \pmod{m}$$

$$y_{t+1} = y_t - \left[\frac{\lambda m}{2\pi}\left(\sin\frac{2\pi}{m}x_{t+1} + 1 - \cos\frac{2\pi}{m}x_{t+1}\right)\right] \pmod{m}$$

lattice size (m)

rounding to nearest integer ($[\,\cdot\,]$)

- All orbits are periodic.
- Orbit representing curves develop some "thickness", but remain stable. why?
- In the chaotic regions the map behaves like a random permutation.

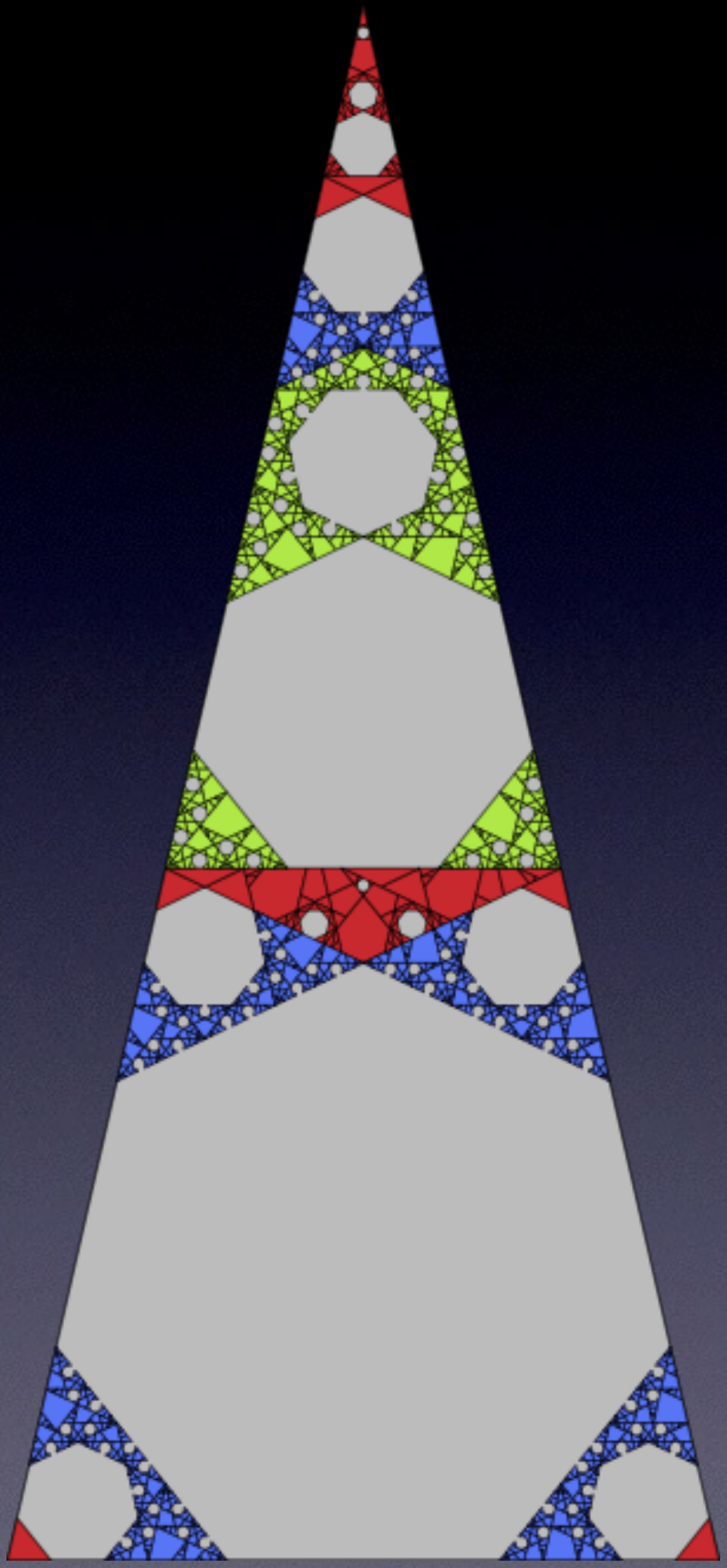


Renormalization

in parametrised families of polygon-exchange transformations

Franco Vivaldi
Queen Mary, University of London

with J H Lowenstein (New York)



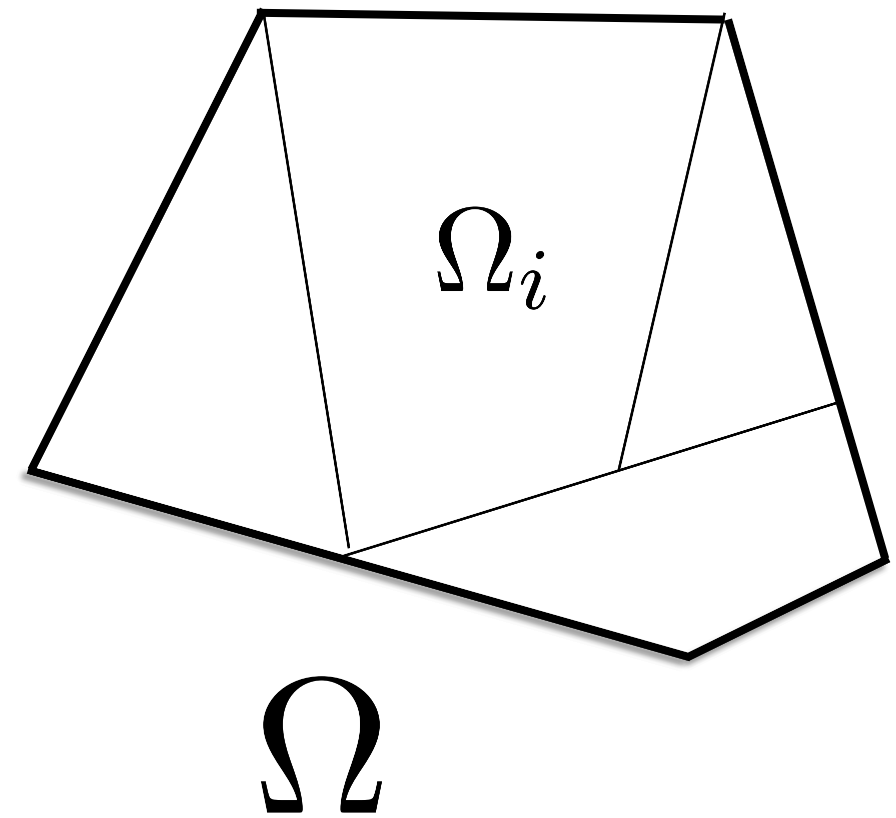
Piecewise isometries

the space:

$$\Omega \subset \mathbb{R}^n$$

$$\Omega = \overline{\bigcup \Omega_i}$$

a finite collection of pairwise disjoint open polytopes (intersection of open half-spaces), called the atoms.

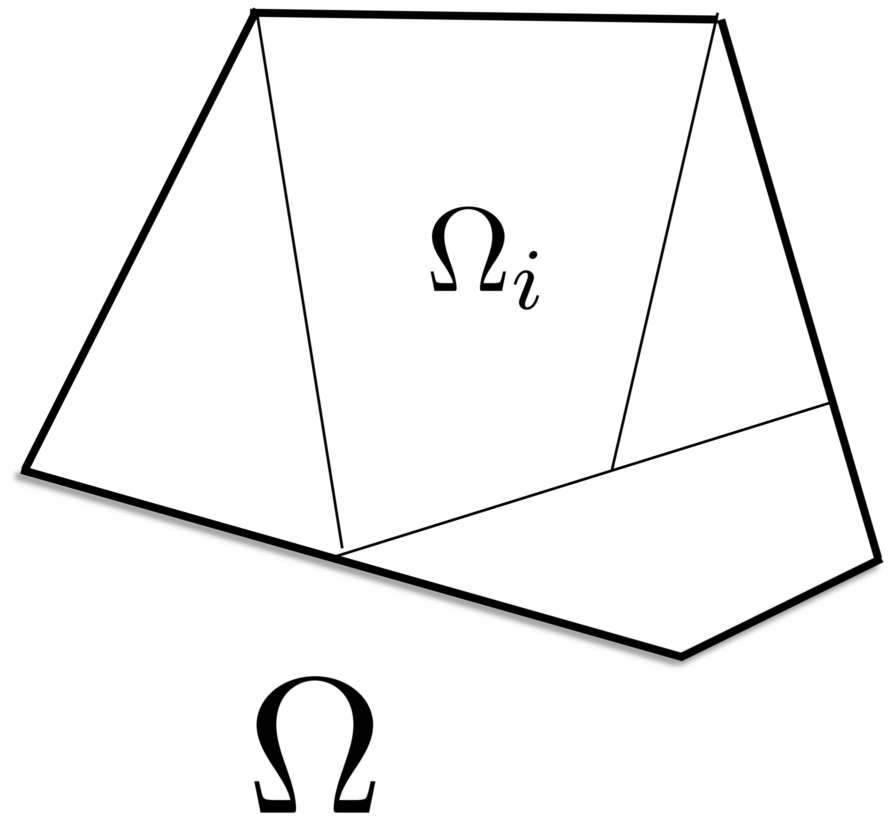


Piecewise isometries

the space:

$$\Omega \subset \mathbb{R}^n$$

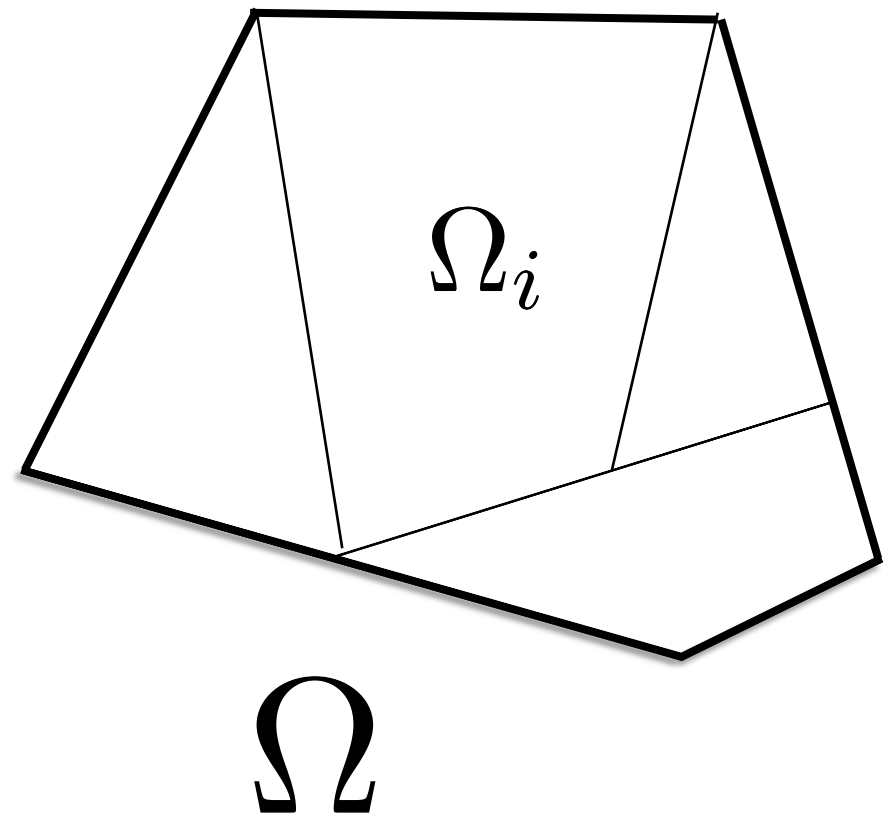
$$\Omega = \overline{\bigcup \Omega_i}$$



a finite collection of pairwise disjoint open polytopes (intersection of open half-spaces), called the atoms.

the dynamics: $F : \Omega \to \Omega$ $\quad F|_{\Omega_i}$ is an isometry

Piecewise isometries



the space:

$$\Omega \subset \mathbb{R}^n$$

$$\Omega = \overline{\bigcup \Omega_i}$$

a finite collection of pairwise disjoint open polytopes (intersection of open half-spaces), called the atoms.

the dynamics: $F : \Omega \to \Omega$ $\quad F|_{\Omega_i}$ is an isometry

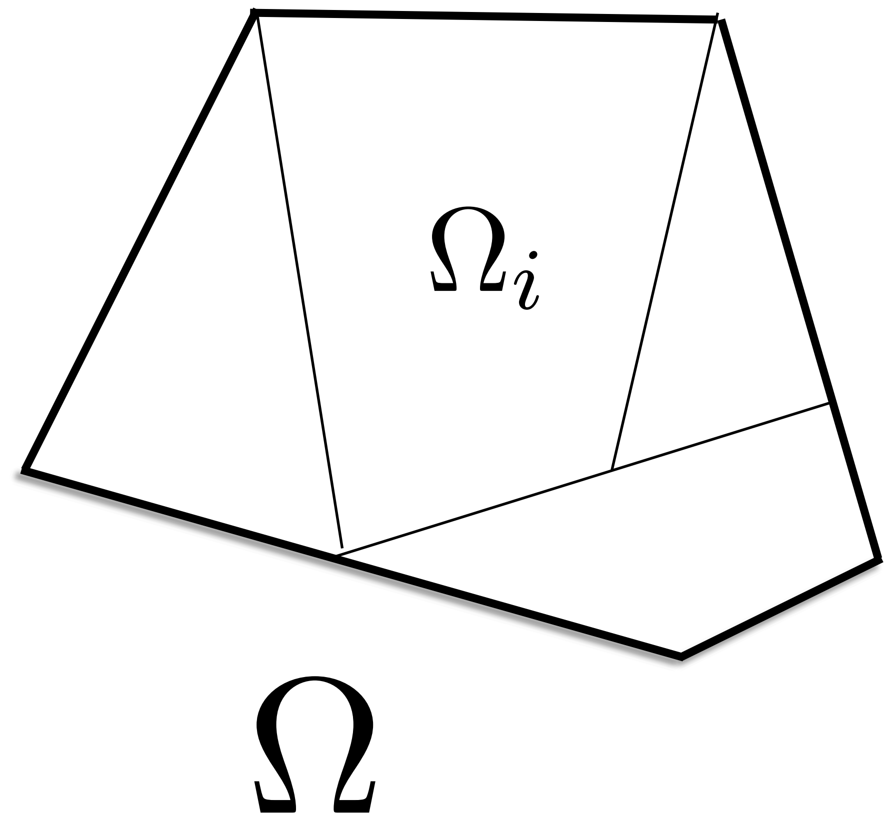
If F is invertible, then F is volume-preserving.

Piecewise isometries

the space:

$$\Omega \subset \mathbb{R}^n$$

$$\Omega = \overline{\bigcup \Omega_i}$$



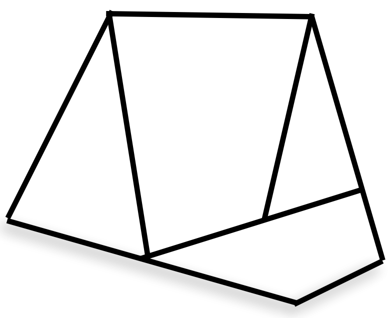
a finite collection of pairwise disjoint open polytopes (intersection of open half-spaces), called the atoms.

the dynamics: $F : \Omega \to \Omega$ $\quad F|_{\Omega_i}$ is an isometry

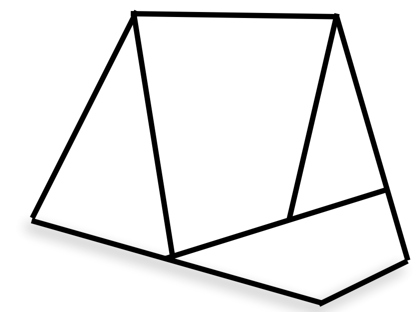
If F is invertible, then F is volume-preserving.

Theorem (Gutkin & Haydin 1997, Buzzi 2001)
The topological entropy of a piecewise isometry is zero.

Higher dimensions: topology

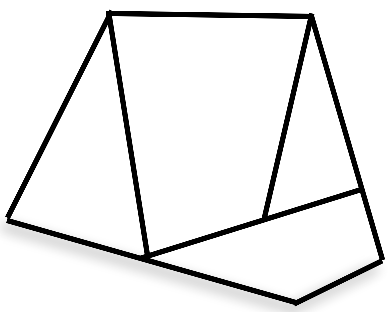


Higher dimensions: topology



Iterate the boundary of the atoms: $\partial\Omega = \bigcup \partial\Omega_i$

Higher dimensions: topology

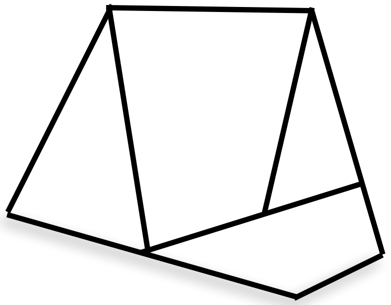


Iterate the boundary of the atoms: $\partial\Omega = \bigcup \partial\Omega_i$

discontinuity set

$$\mathscr{D} = \bigcup_{t\in\mathbb{Z}} F^t(\partial\Omega)$$

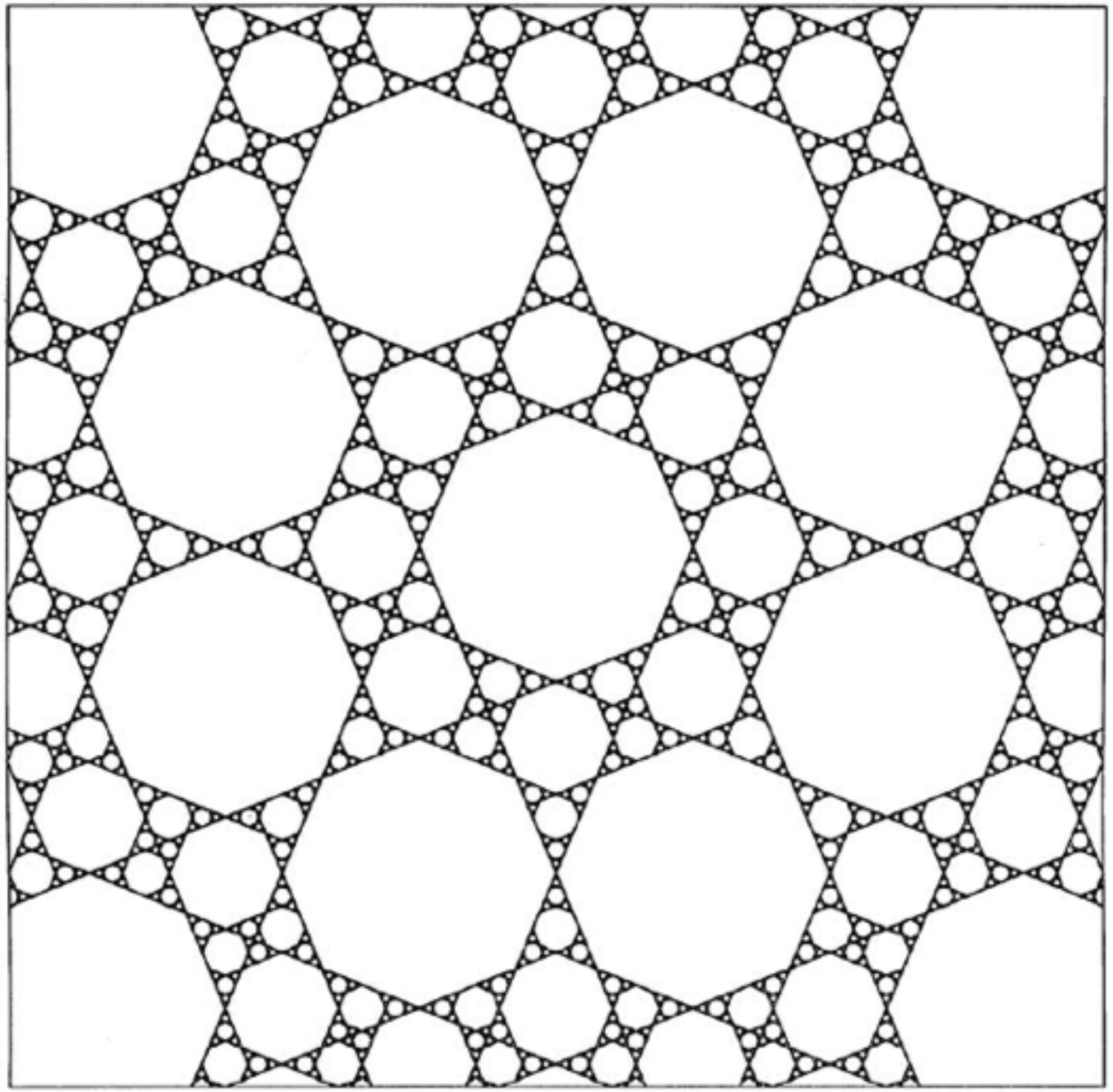
Higher dimensions: topology



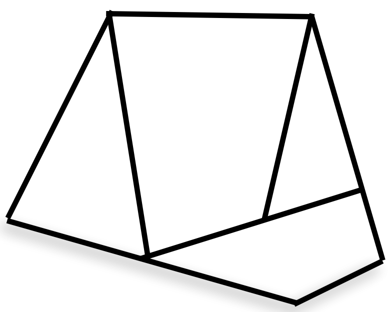
Iterate the boundary of the atoms: $\partial\Omega = \bigcup \partial\Omega_i$

discontinuity set

$$\mathscr{D} = \bigcup_{t\in\mathbb{Z}} F^t(\partial\Omega)$$



Higher dimensions: topology

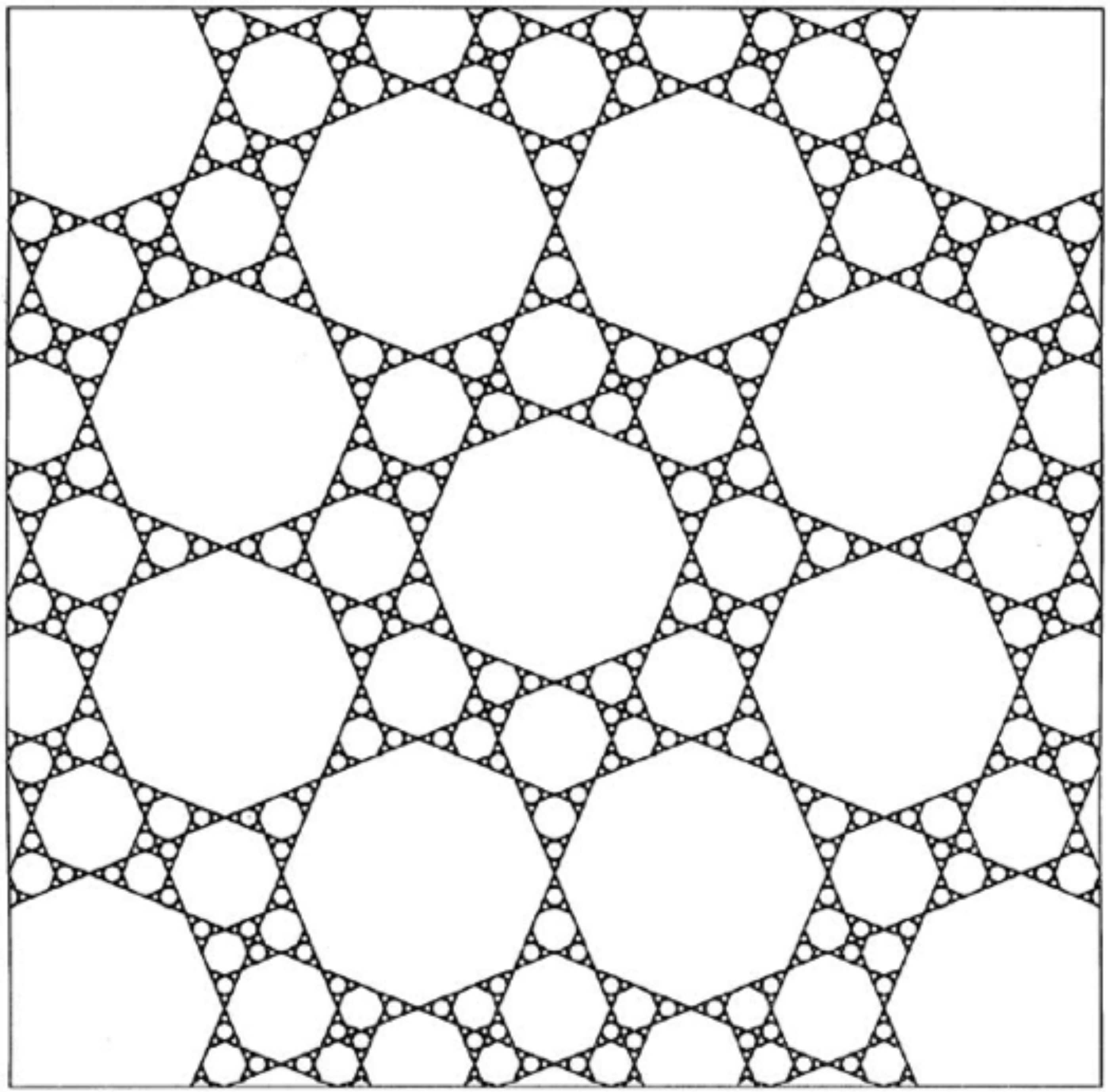


Iterate the boundary of the atoms: $\partial\Omega = \bigcup \partial\Omega_i$

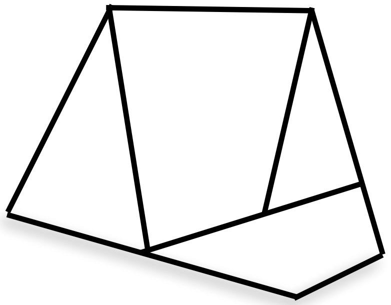
discontinuity set

$$\mathscr{D} = \bigcup_{t\in\mathbb{Z}} F^t(\partial\Omega)$$

not dense, typically



Higher dimensions: topology

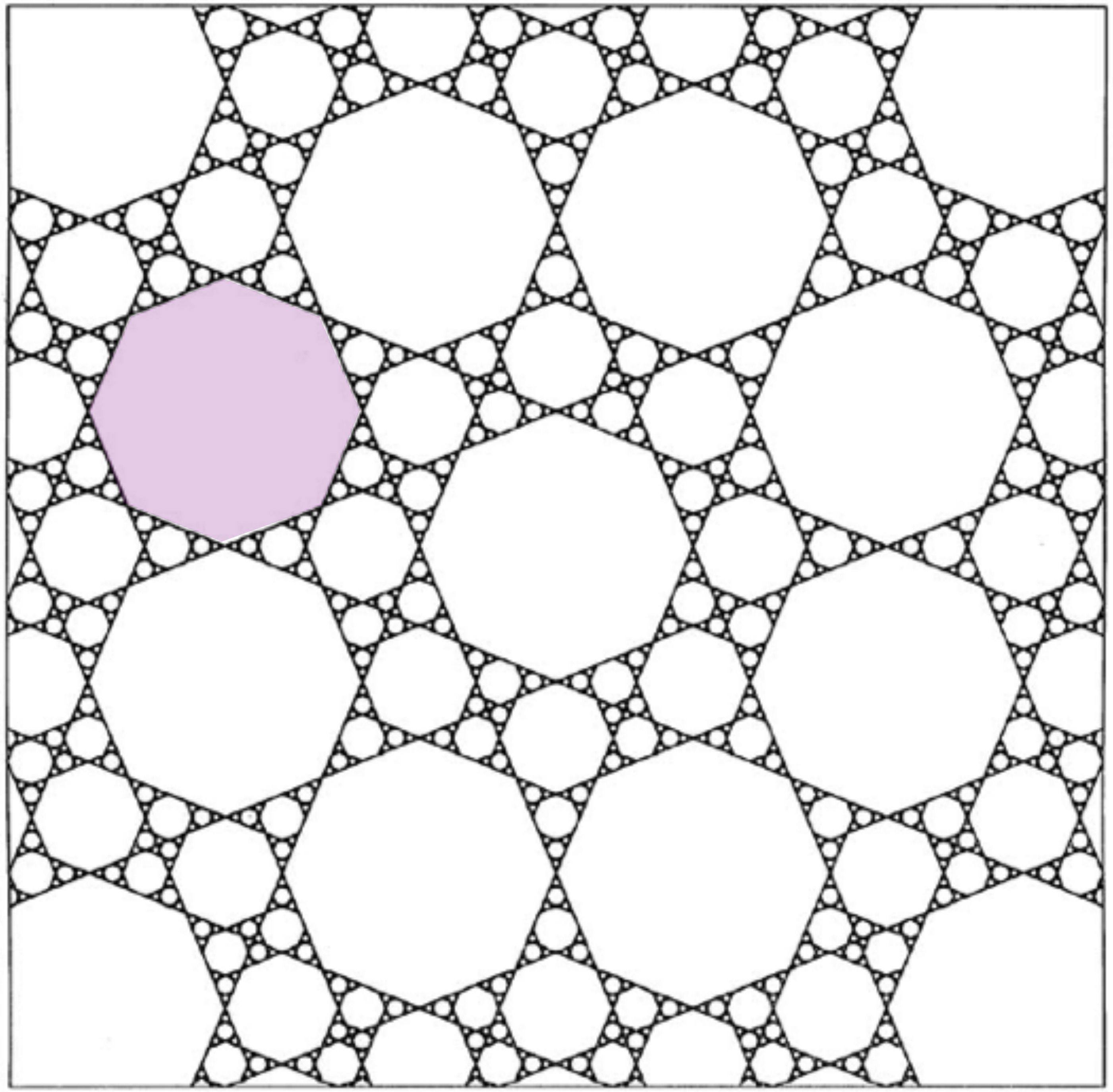


Iterate the boundary of the atoms: $\partial\Omega = \bigcup \partial\Omega_i$

discontinuity set

$$\mathscr{D} = \bigcup_{t\in\mathbb{Z}} F^t(\partial\Omega)$$

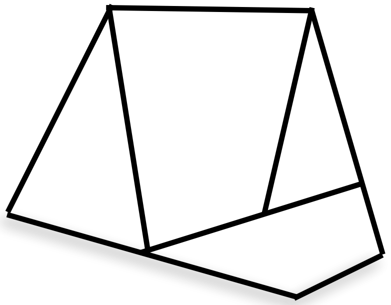
not dense, typically



periodic set

$$\Pi = \Omega \setminus \overline{\mathscr{D}}$$

Higher dimensions: topology



Iterate the boundary of the atoms: $\partial\Omega = \bigcup \partial\Omega_i$

discontinuity set

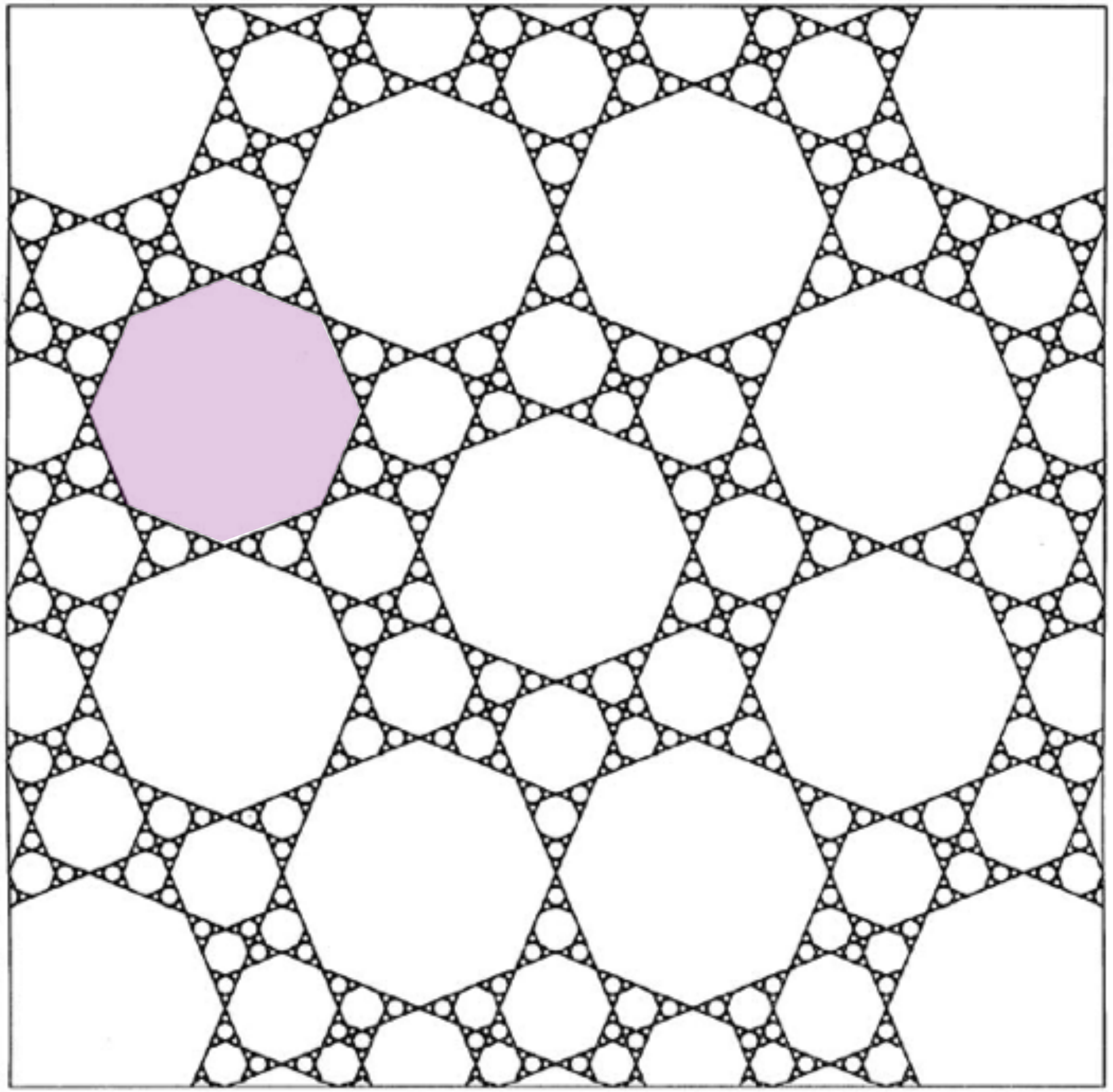
$$\mathscr{D} = \bigcup_{t\in\mathbb{Z}} F^t(\partial\Omega)$$

not dense, typically

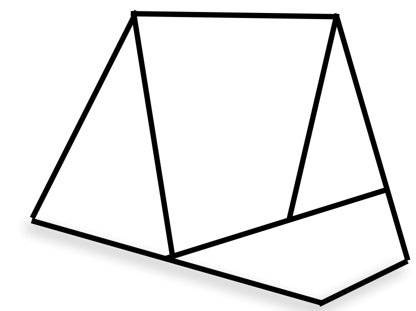
periodic set

$$\Pi = \Omega \setminus \overline{\mathscr{D}}$$

(union of cells of positive measure)



Higher dimensions: topology

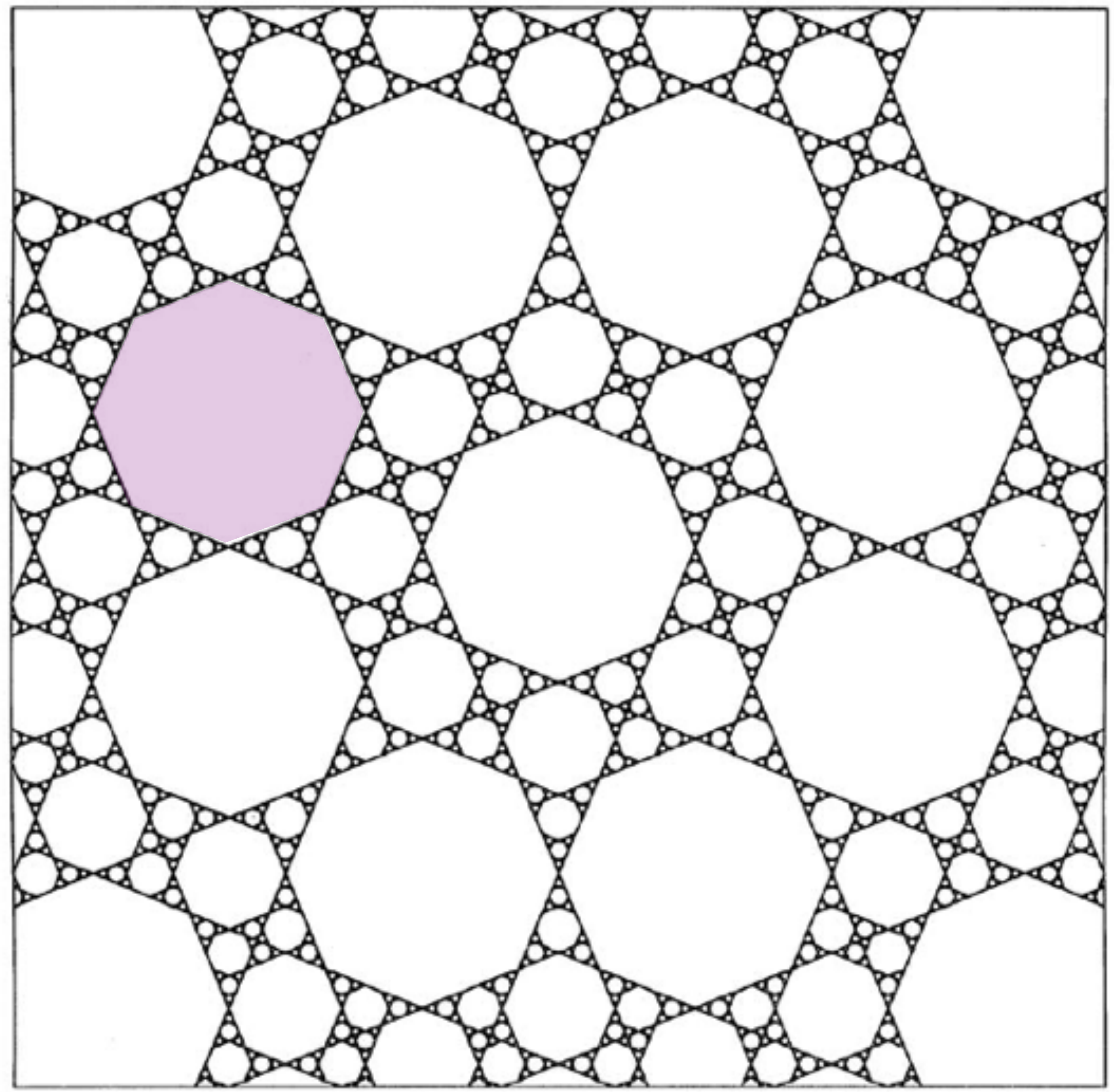


Iterate the boundary of the atoms: $\partial\Omega = \bigcup \partial\Omega_i$

discontinuity set

$$\mathscr{D} = \bigcup_{t\in\mathbb{Z}} F^t(\partial\Omega)$$

not dense, typically



periodic set

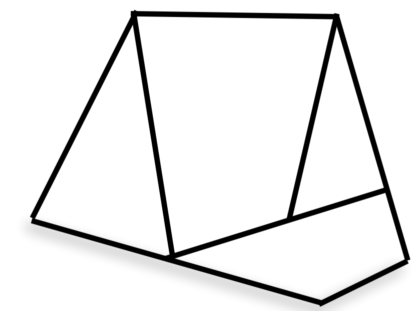
$$\Pi = \Omega \setminus \overline{\mathscr{D}}$$

(union of cells of positive measure)

exceptional set

$$\mathscr{E} = \overline{\mathscr{D}} \setminus \mathscr{D}$$

Higher dimensions: topology



Iterate the boundary of the atoms: $\partial\Omega = \bigcup \partial\Omega_i$

discontinuity set

$$\mathscr{D} = \bigcup_{t\in\mathbb{Z}} F^t(\partial\Omega)$$

not dense, typically

periodic set

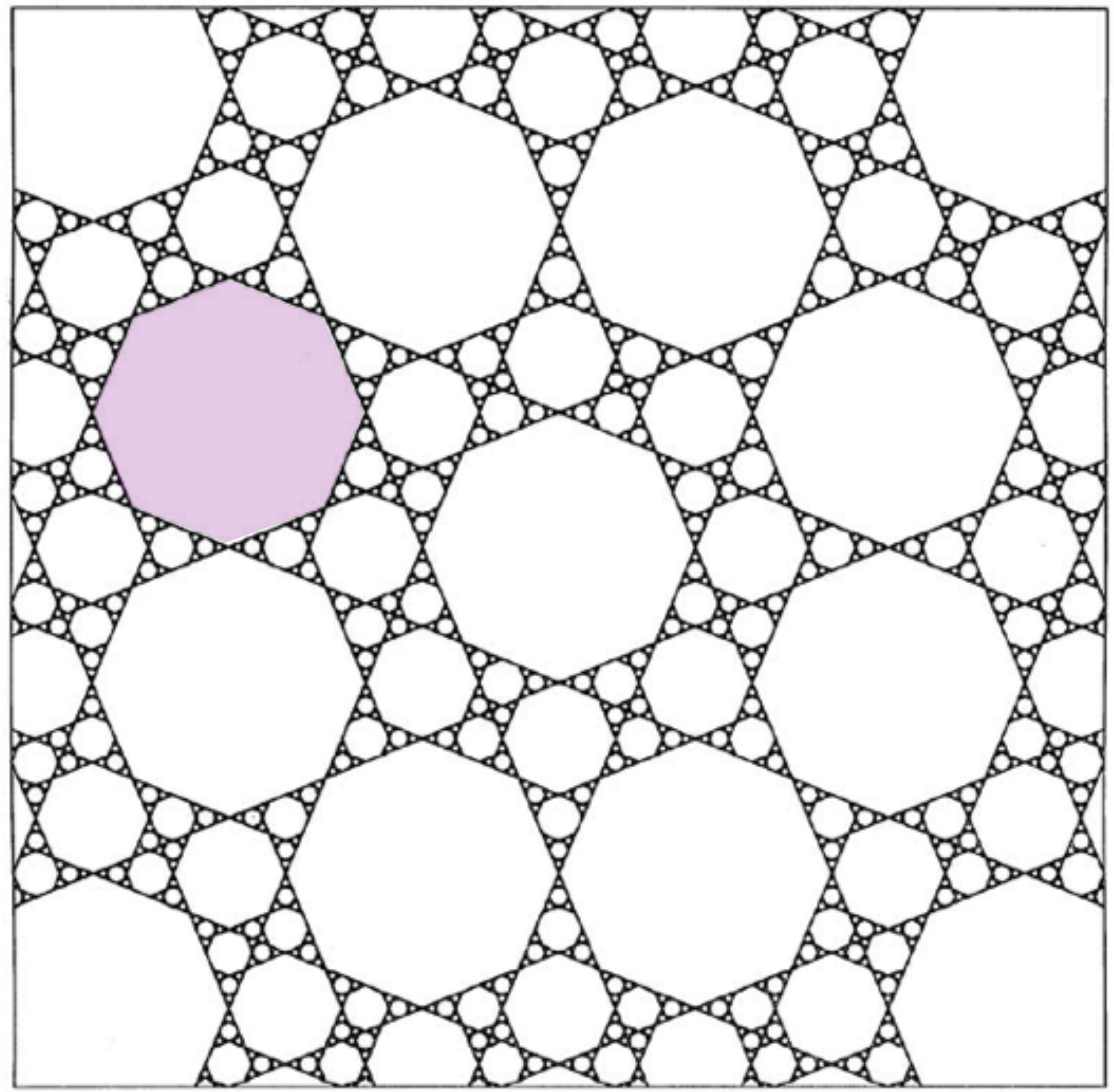
$$\Pi = \Omega \setminus \overline{\mathscr{D}}$$

(union of cells of positive measure)

exceptional set

$$\mathscr{E} = \overline{\mathscr{D}} \setminus \mathscr{D}$$

(asymptotic phenomena)

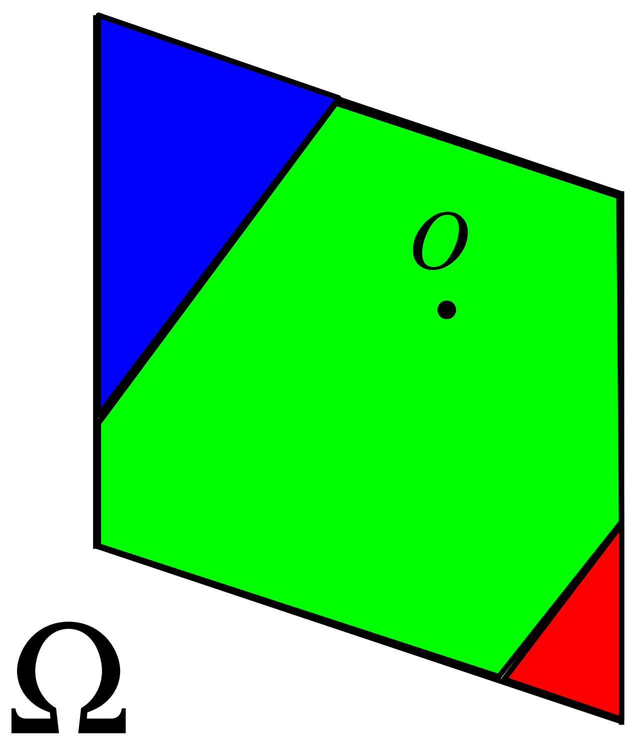


One-parameter families of polygon-exchange transformations

- Hooper (2013)
- Schwartz (2014)
- Lowenstein & fv (2016)

One-parameter families of polygon-exchange transformations

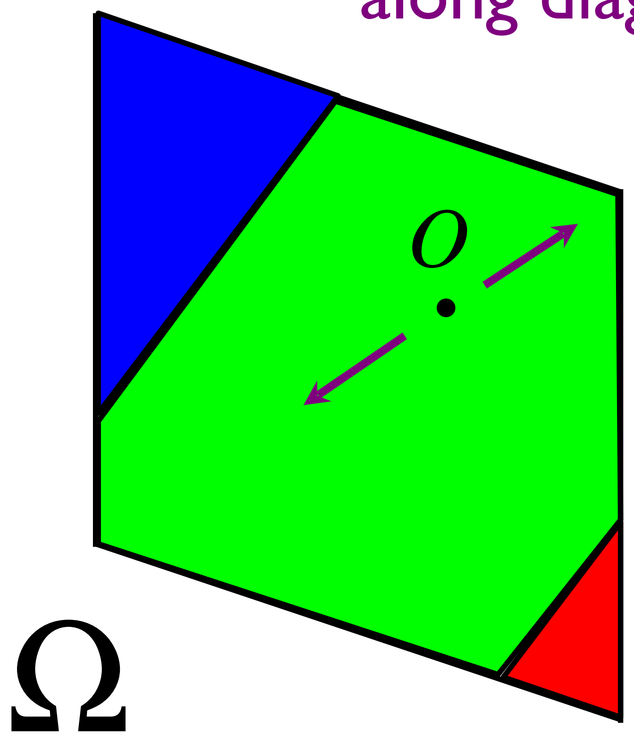
- Hooper (2013)
- Schwartz (2014)
- Lowenstein & fv (2016)



One-parameter families of polygon-exchange transformations

- Hooper (2013)
- Schwartz (2014)
- Lowenstein & fv (2016)

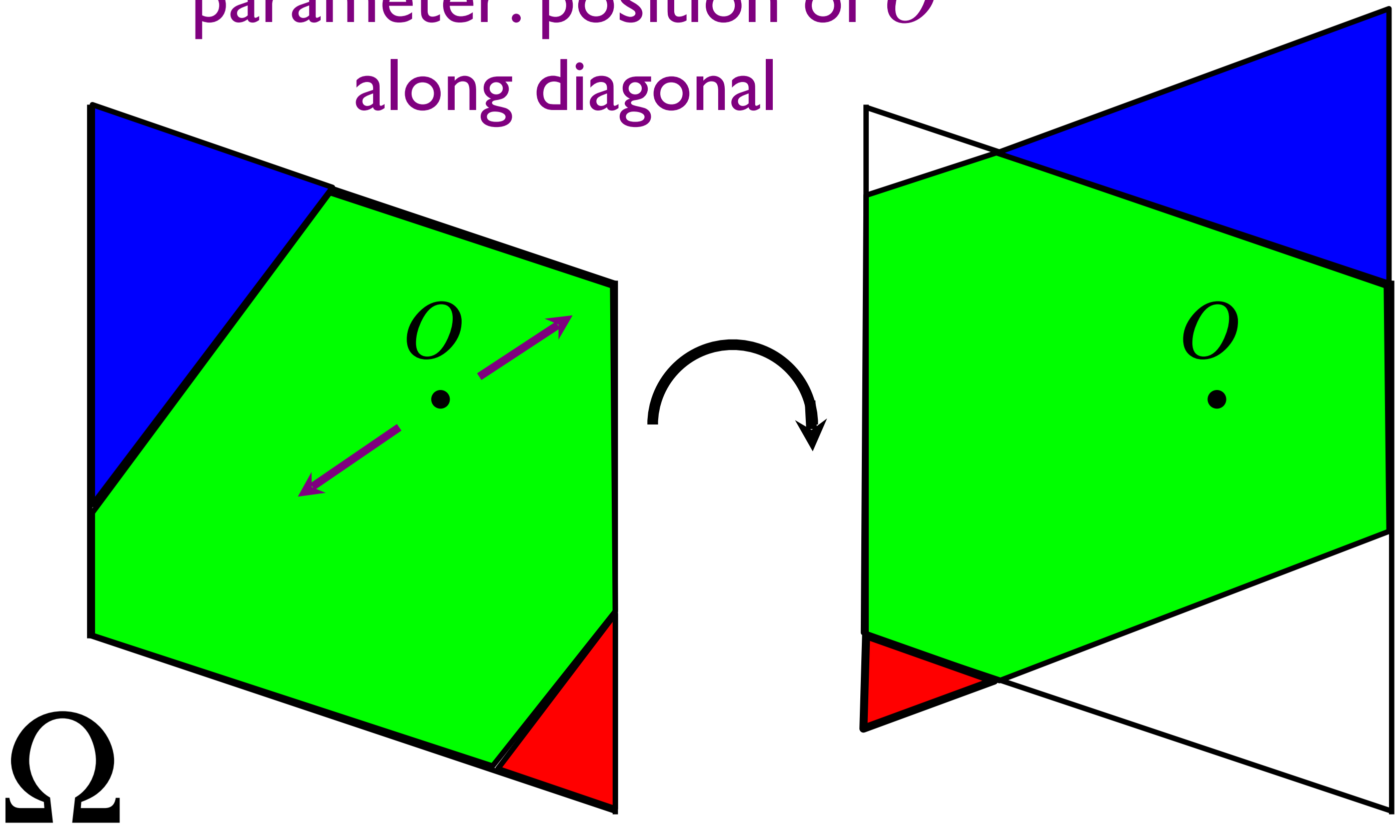
parameter: position of O along diagonal



One-parameter families of polygon-exchange transformations

- Hooper (2013)
- Schwartz (2014)
- Lowenstein & fv (2016)

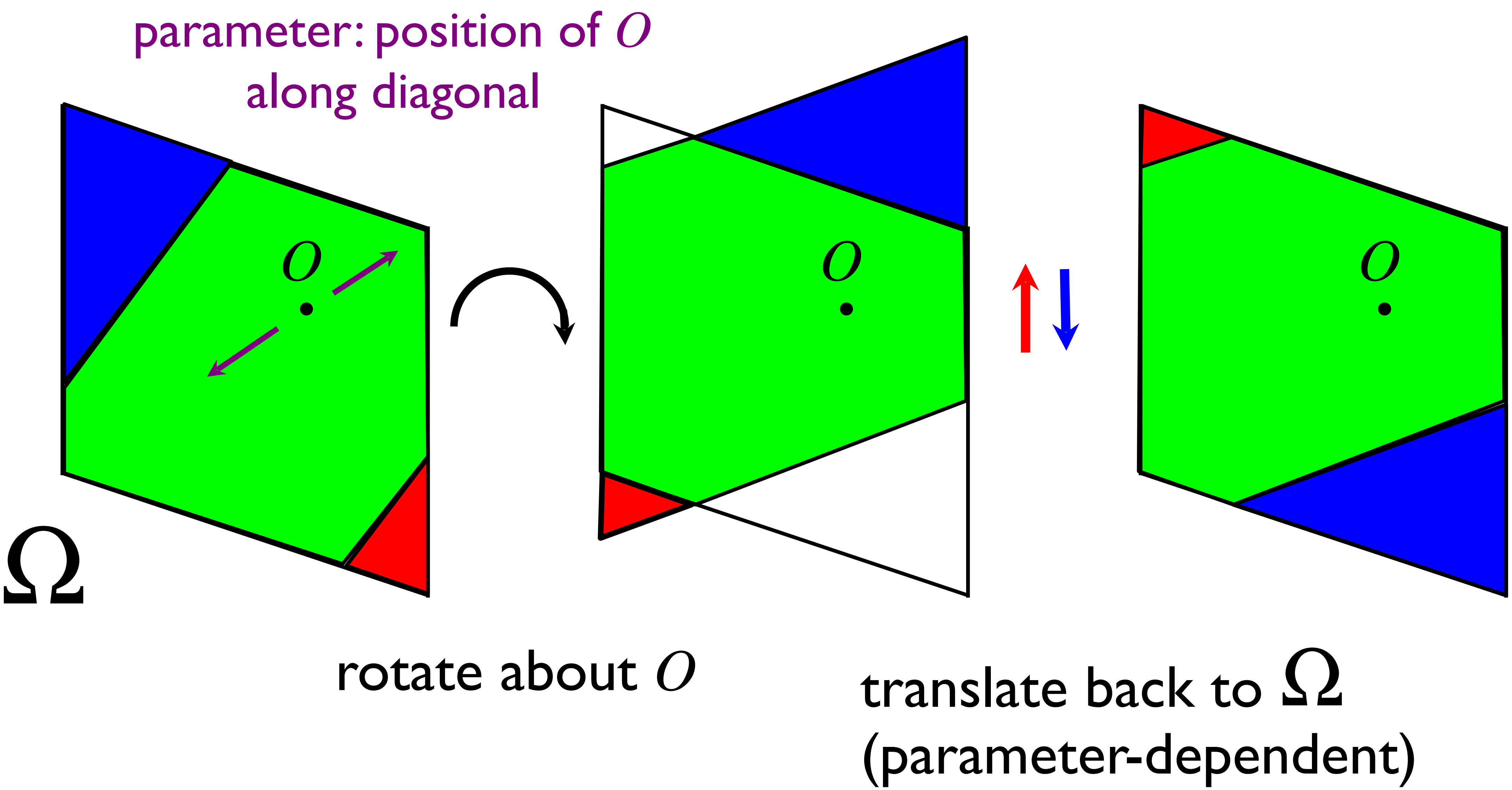
parameter: position of O along diagonal



rotate about O

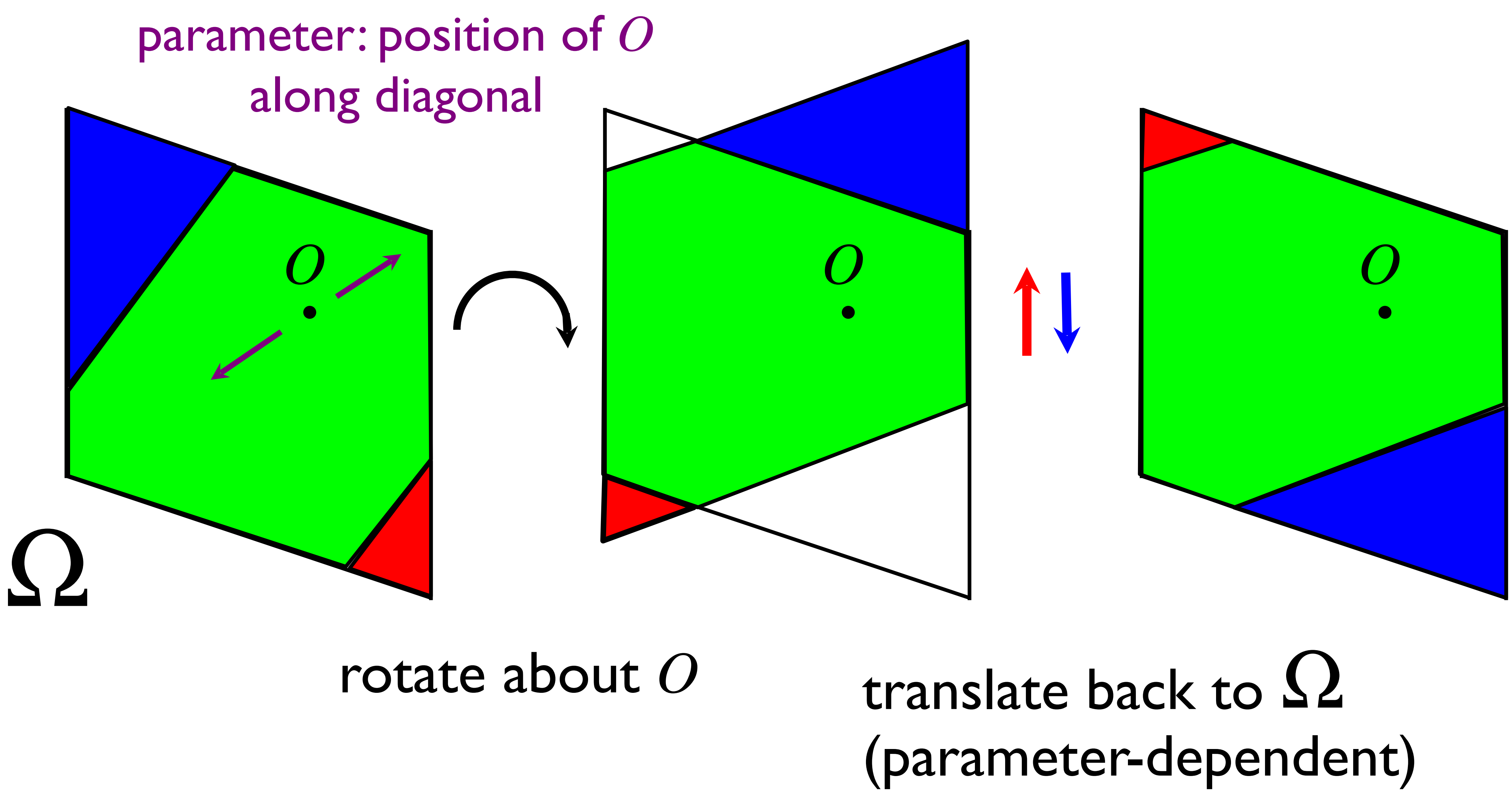
One-parameter families of polygon-exchange transformations

- Hooper (2013)
- Schwartz (2014)
- Lowenstein & fv (2016)



One-parameter families of polygon-exchange transformations

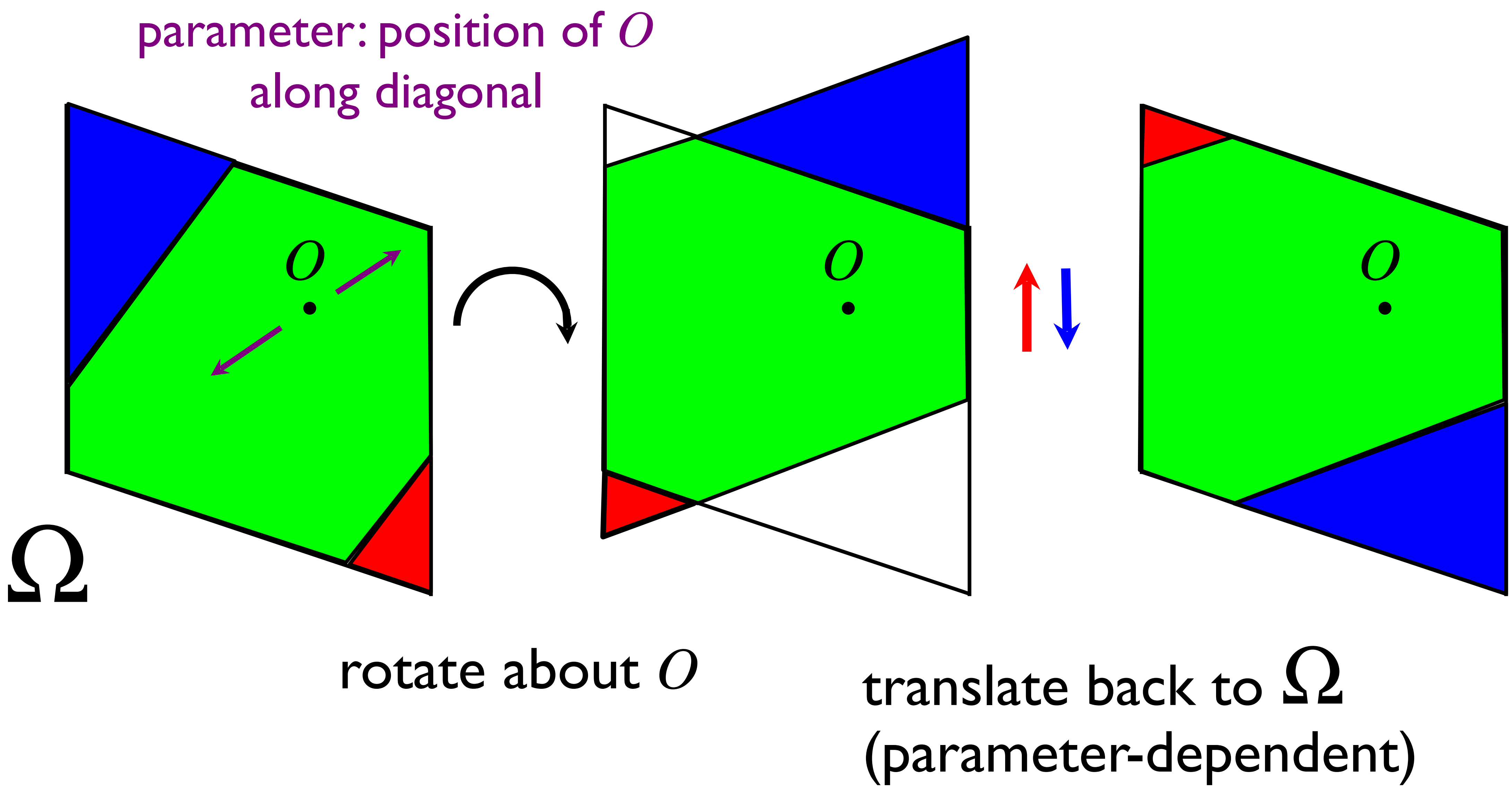
- Hooper (2013)
- Schwartz (2014)
- Lowenstein & fv (2016)



quadratic rotation fields: $\mathbb{Q}(\lambda) = \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{5})$

One-parameter families of polygon-exchange transformations

- Hooper (2013)
- Schwartz (2014)
- Lowenstein & fv (2016)

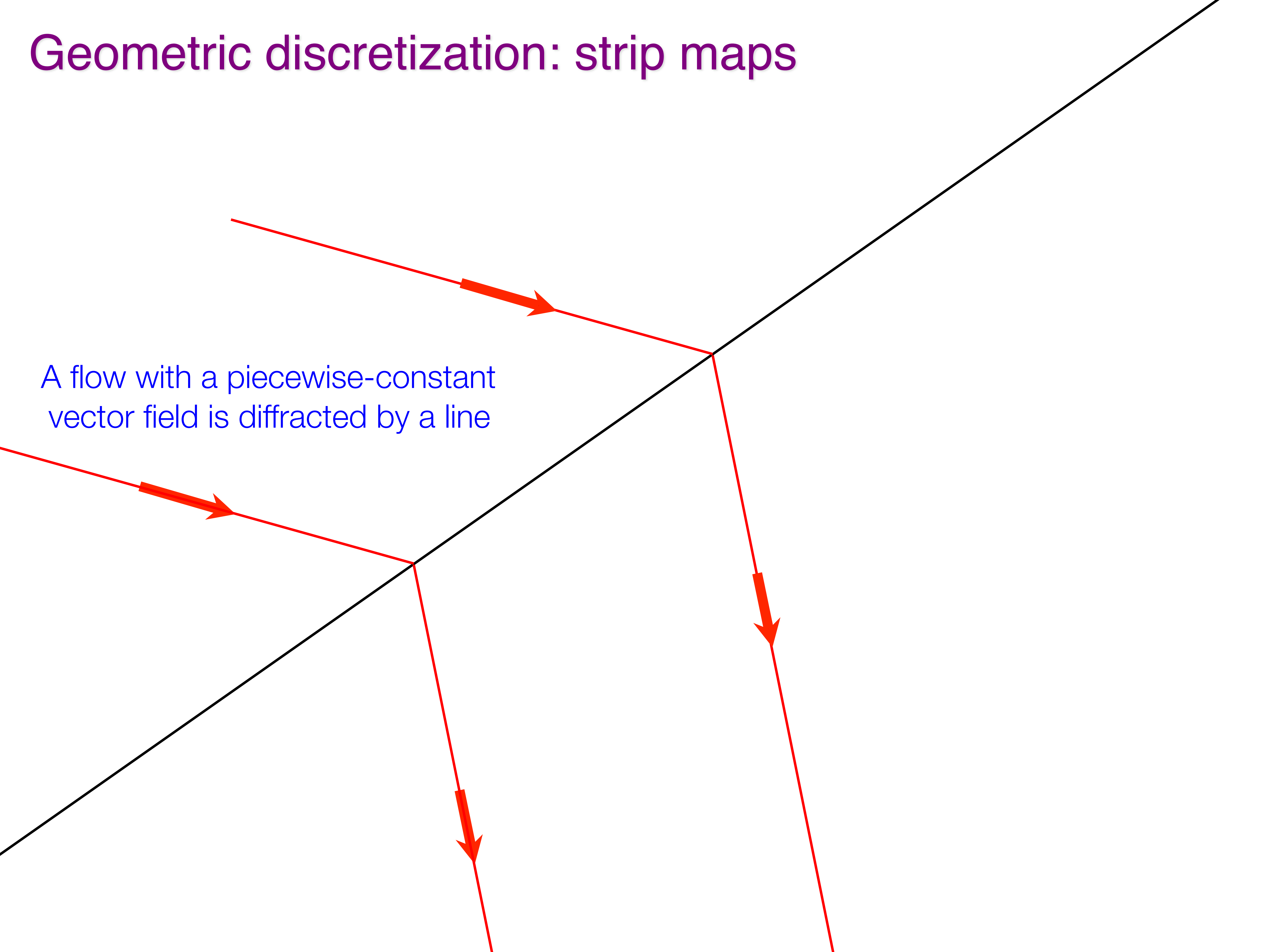


quadratic rotation fields: $\mathbb{Q}(\lambda) = \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{5})$

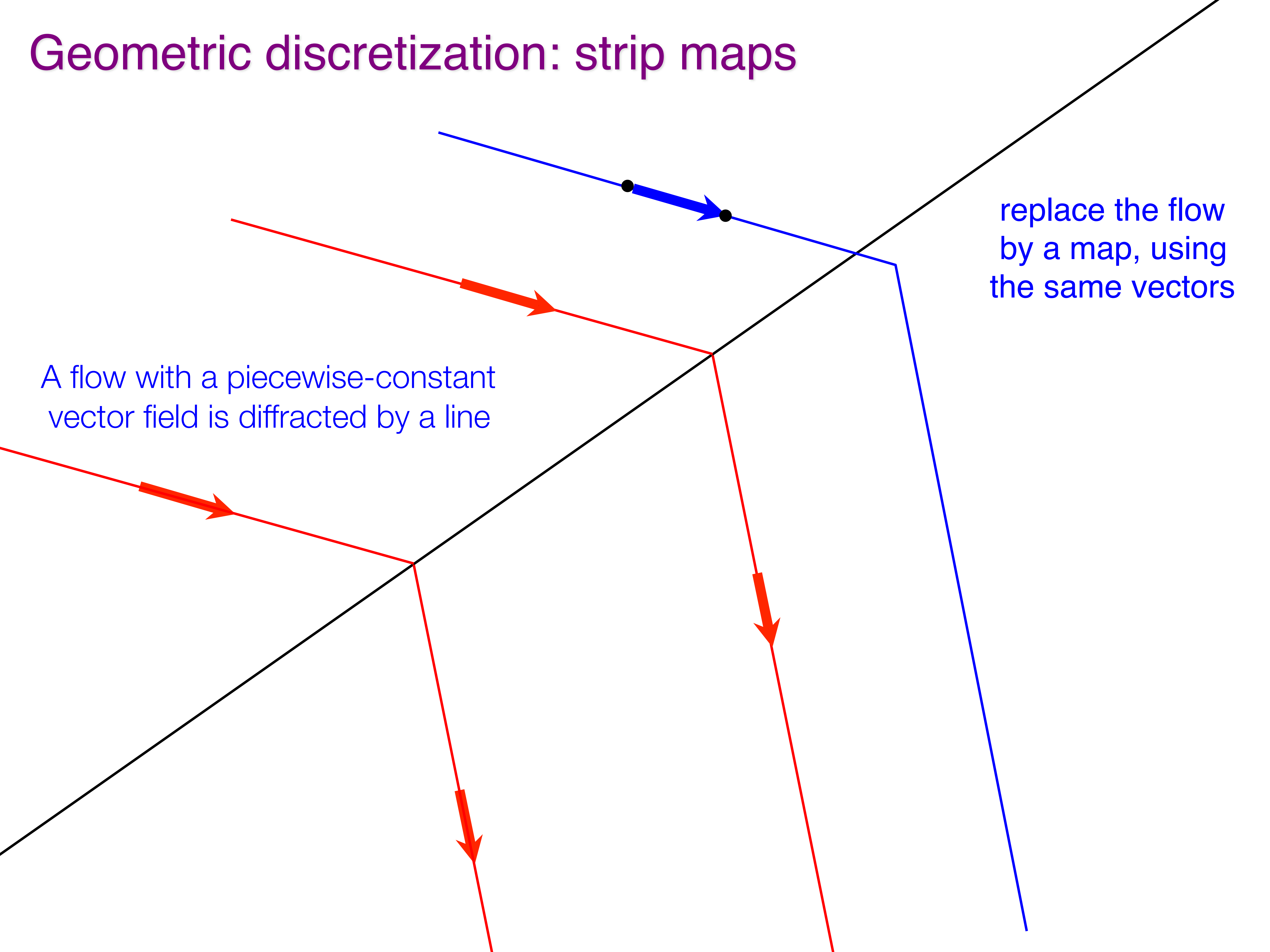
translation module: $\mathbb{Q}(\lambda) + s\,\mathbb{Q}(\lambda)$ $\quad$ s: parameter

Geometric discretization: strip maps

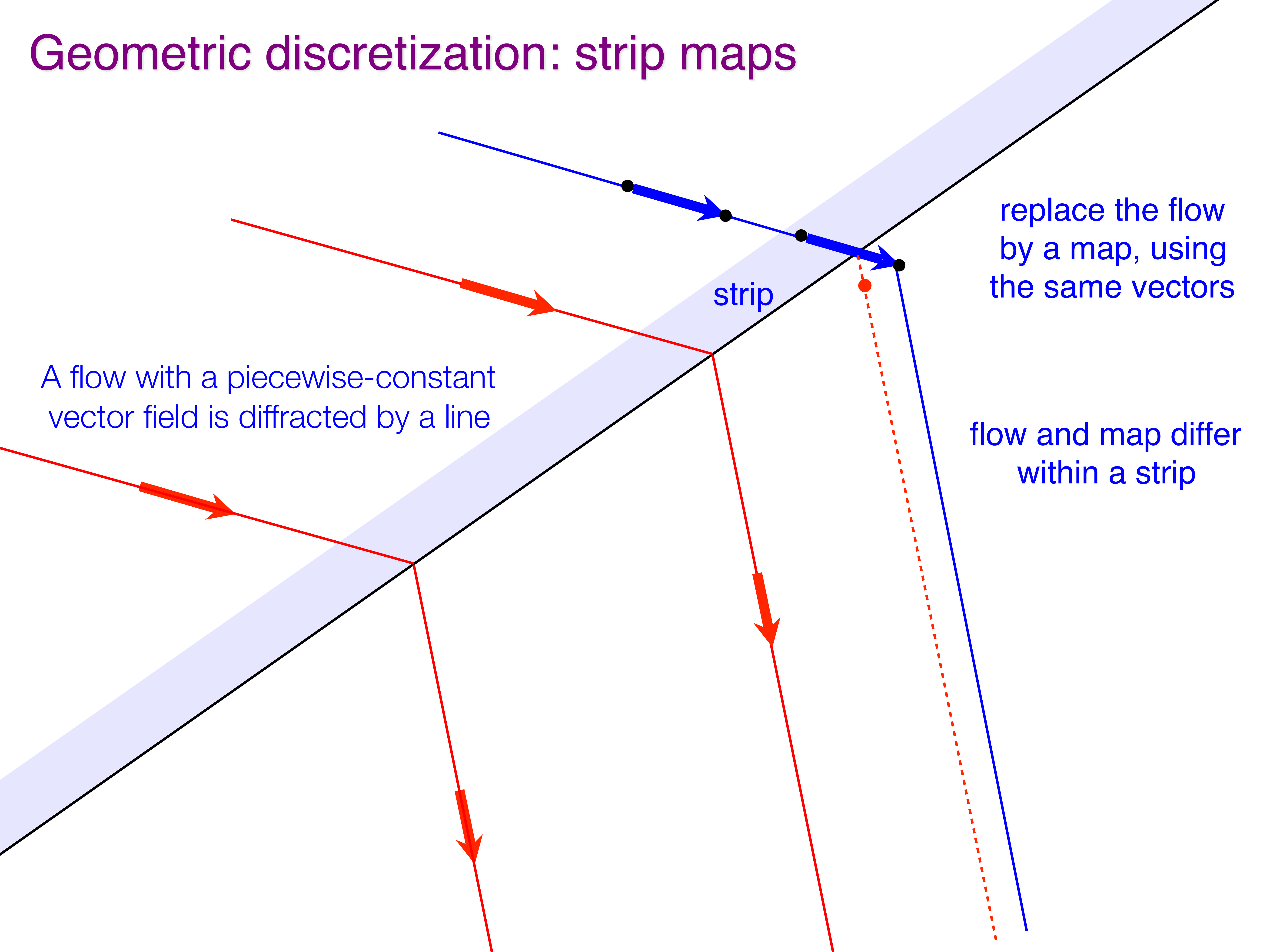
Geometric discretization: strip maps



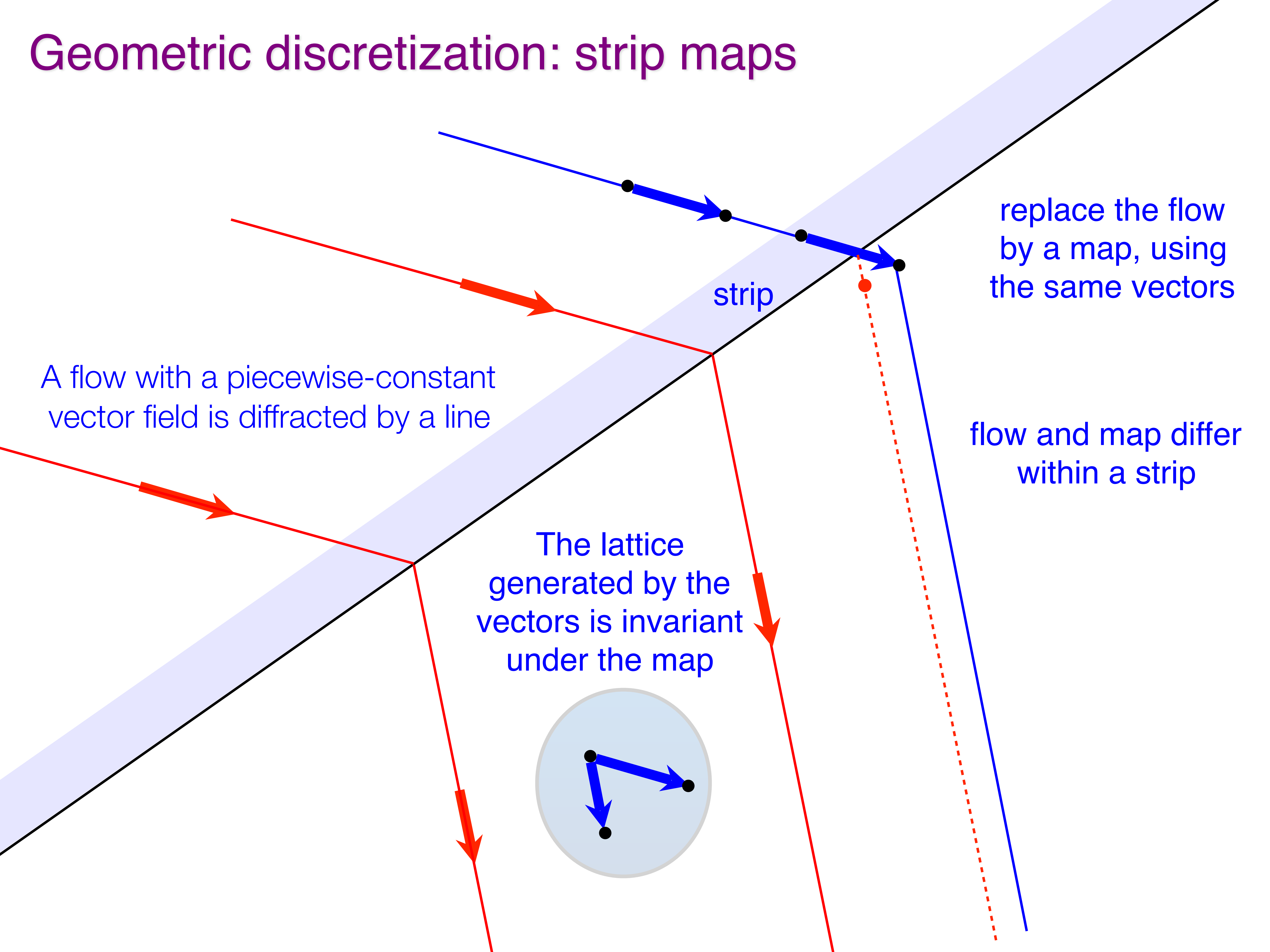
Geometric discretization: strip maps



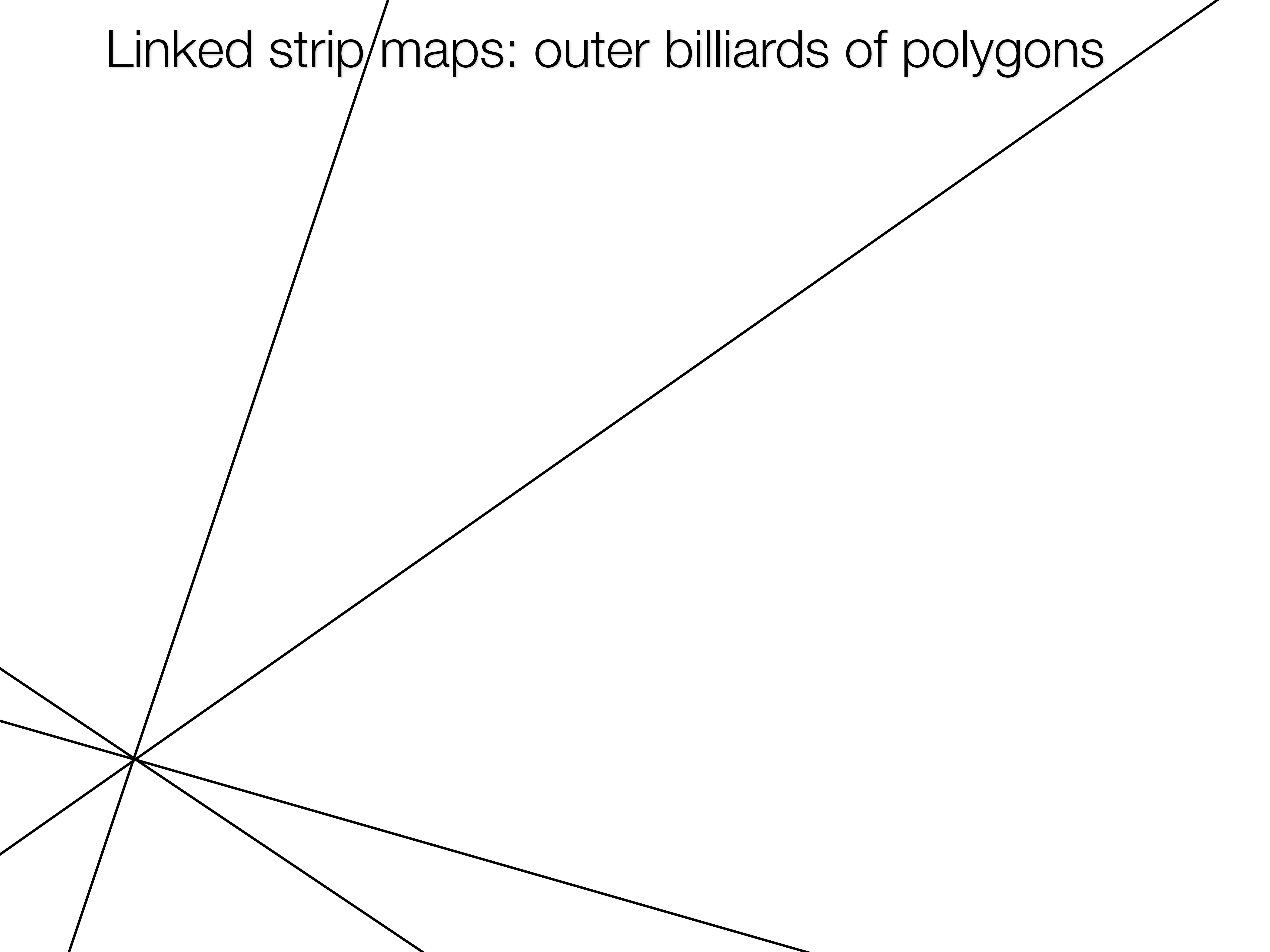
Geometric discretization: strip maps



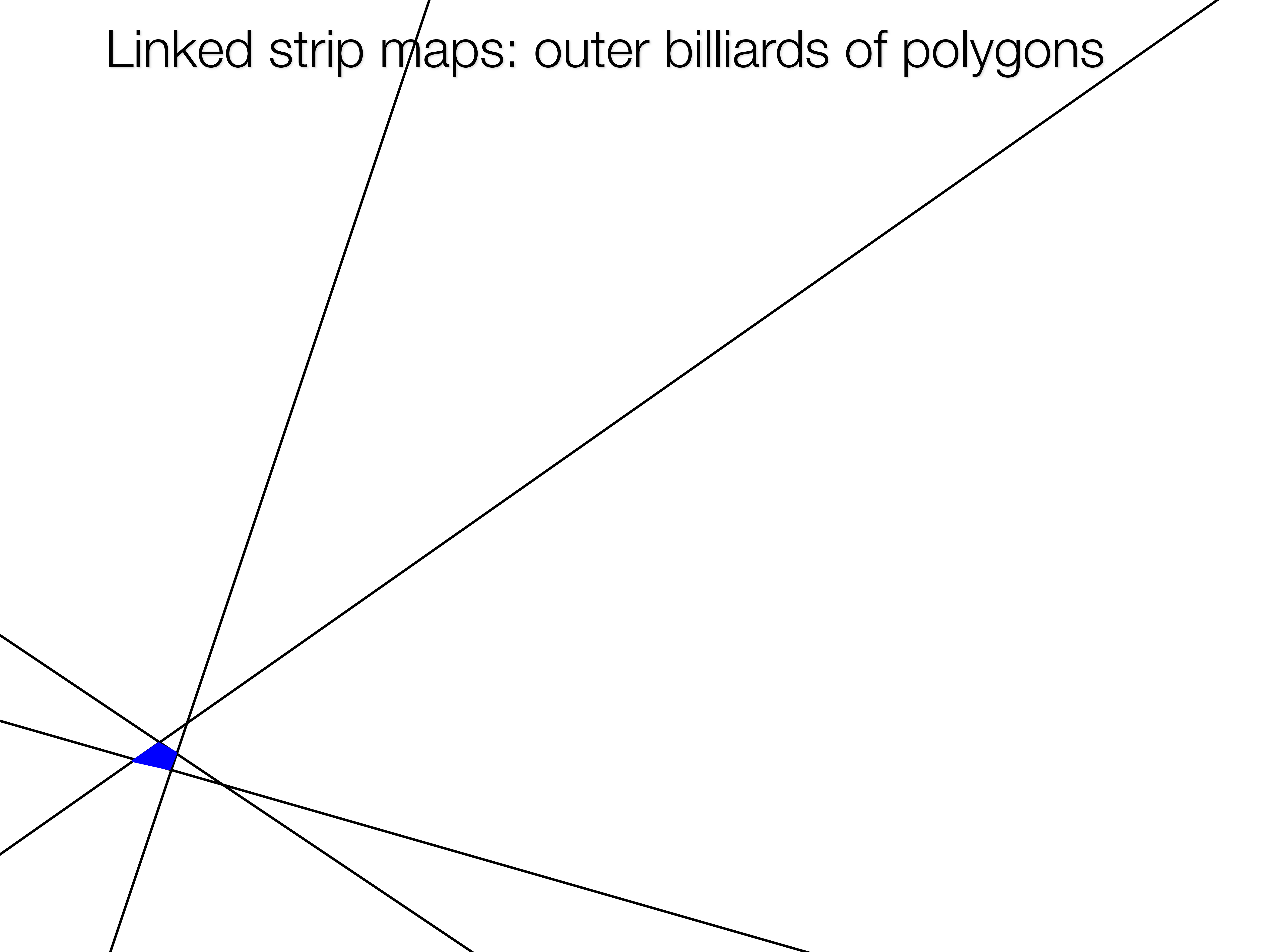
Geometric discretization: strip maps



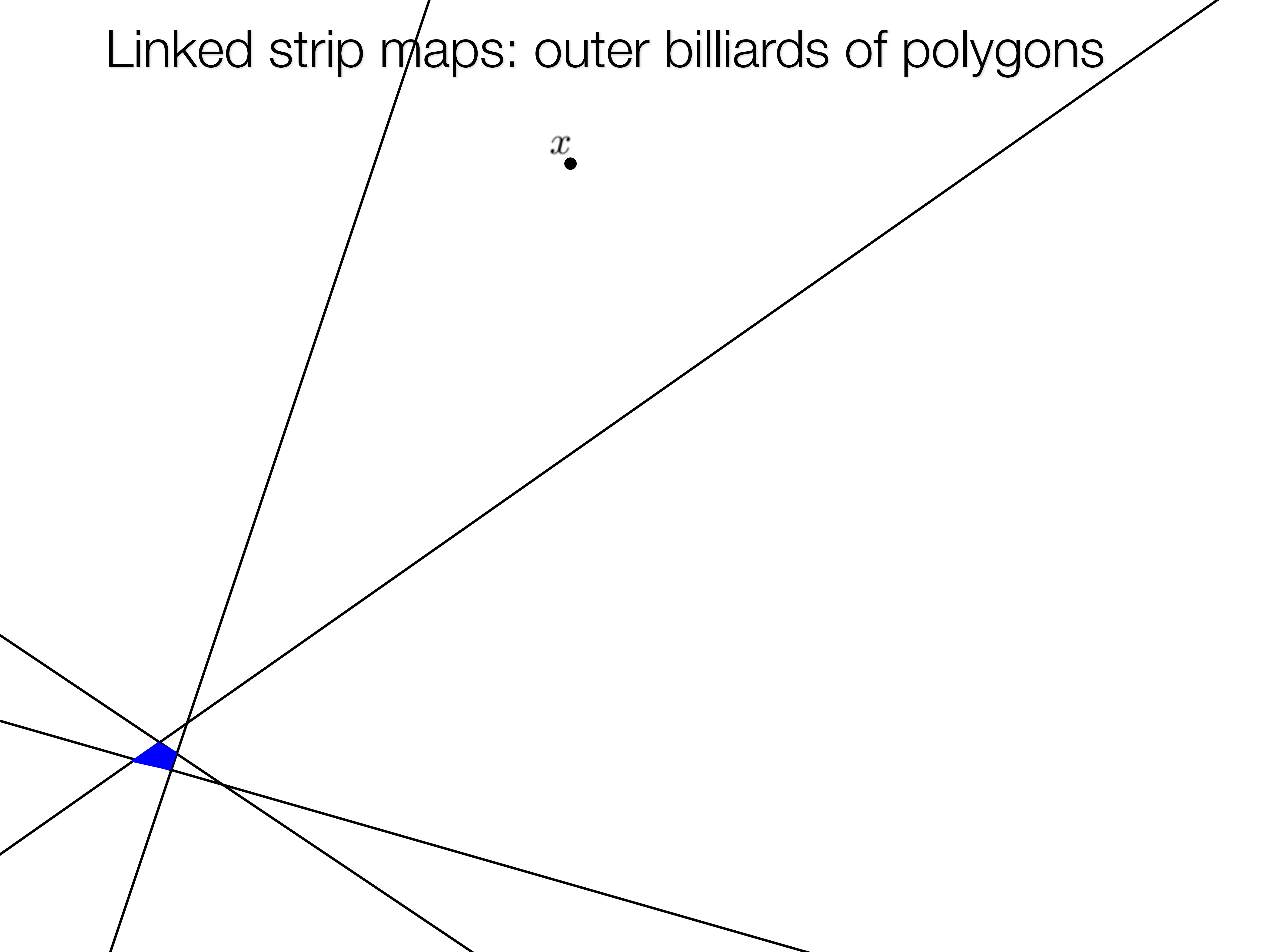
Linked strip maps: outer billiards of polygons



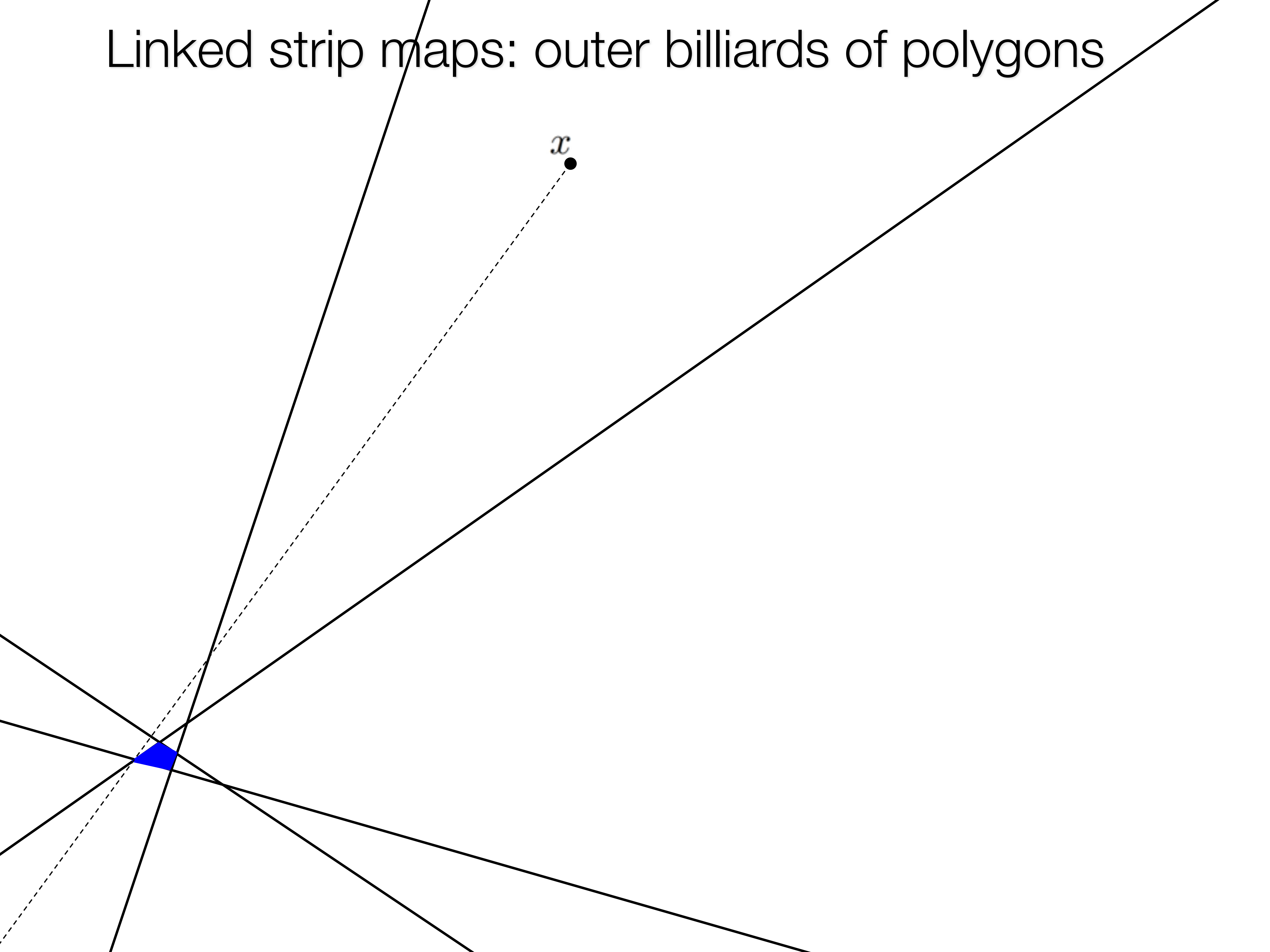
Linked strip maps: outer billiards of polygons



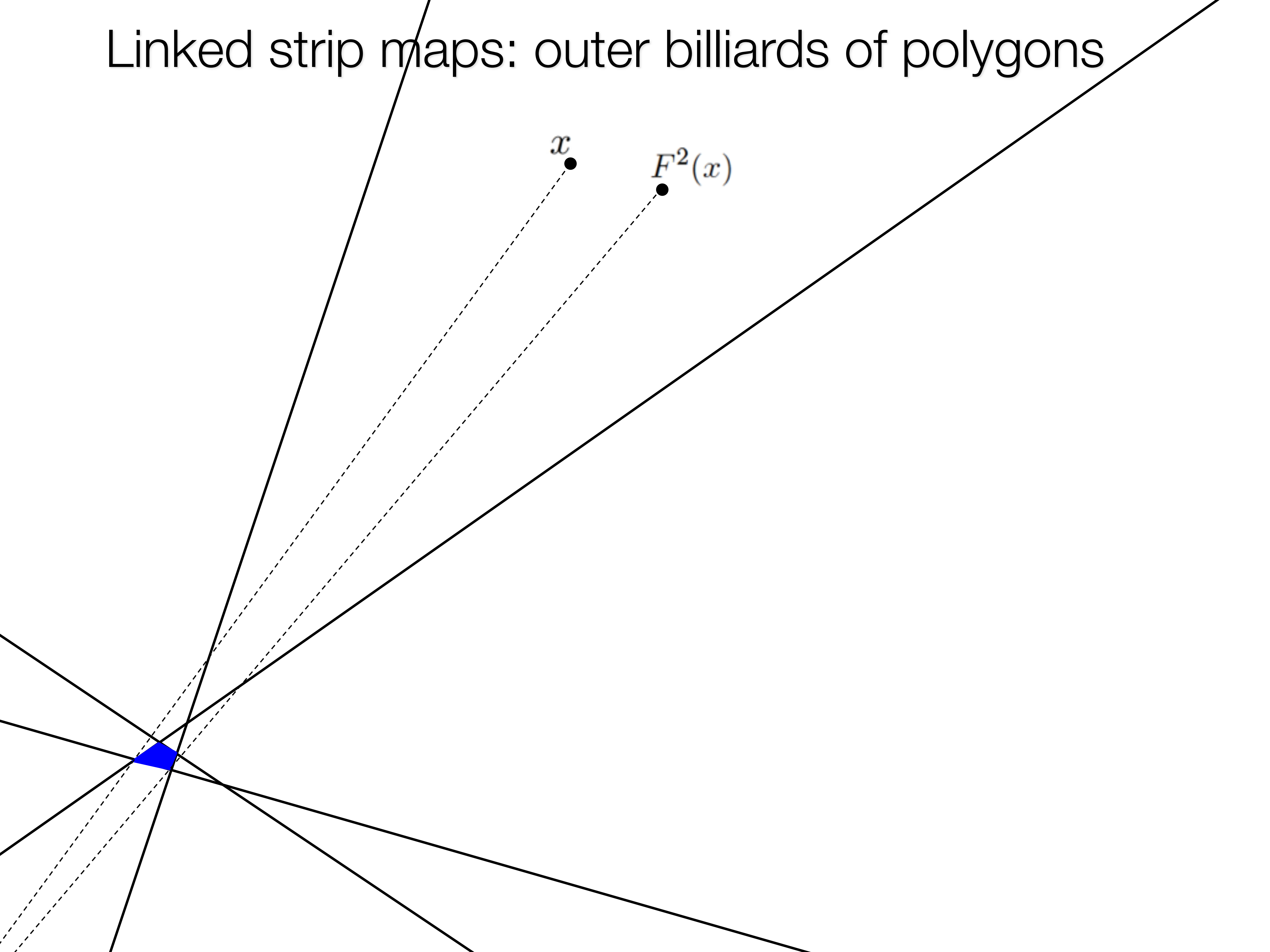
Linked strip maps: outer billiards of polygons



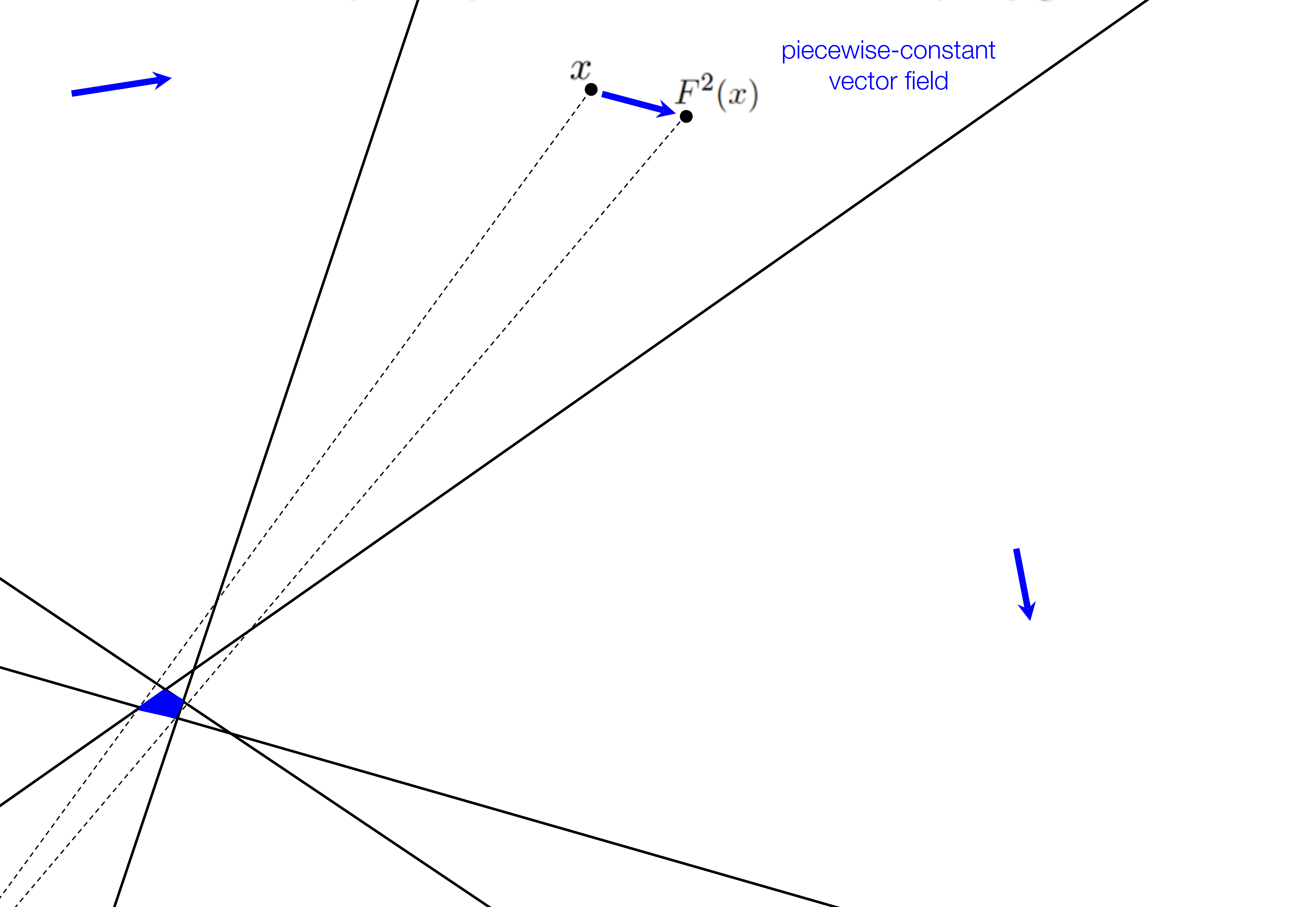
Linked strip maps: outer billiards of polygons



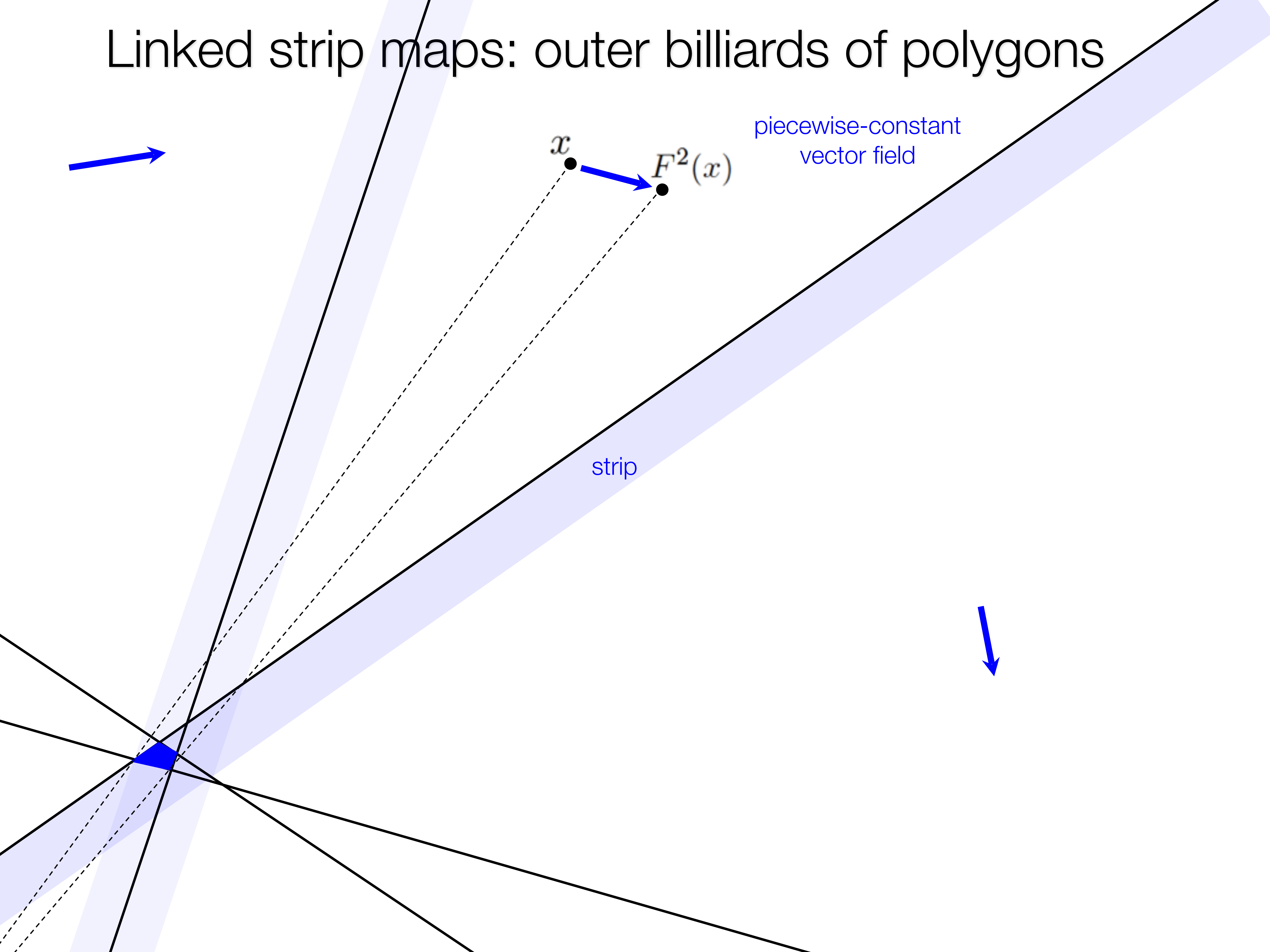
Linked strip maps: outer billiards of polygons



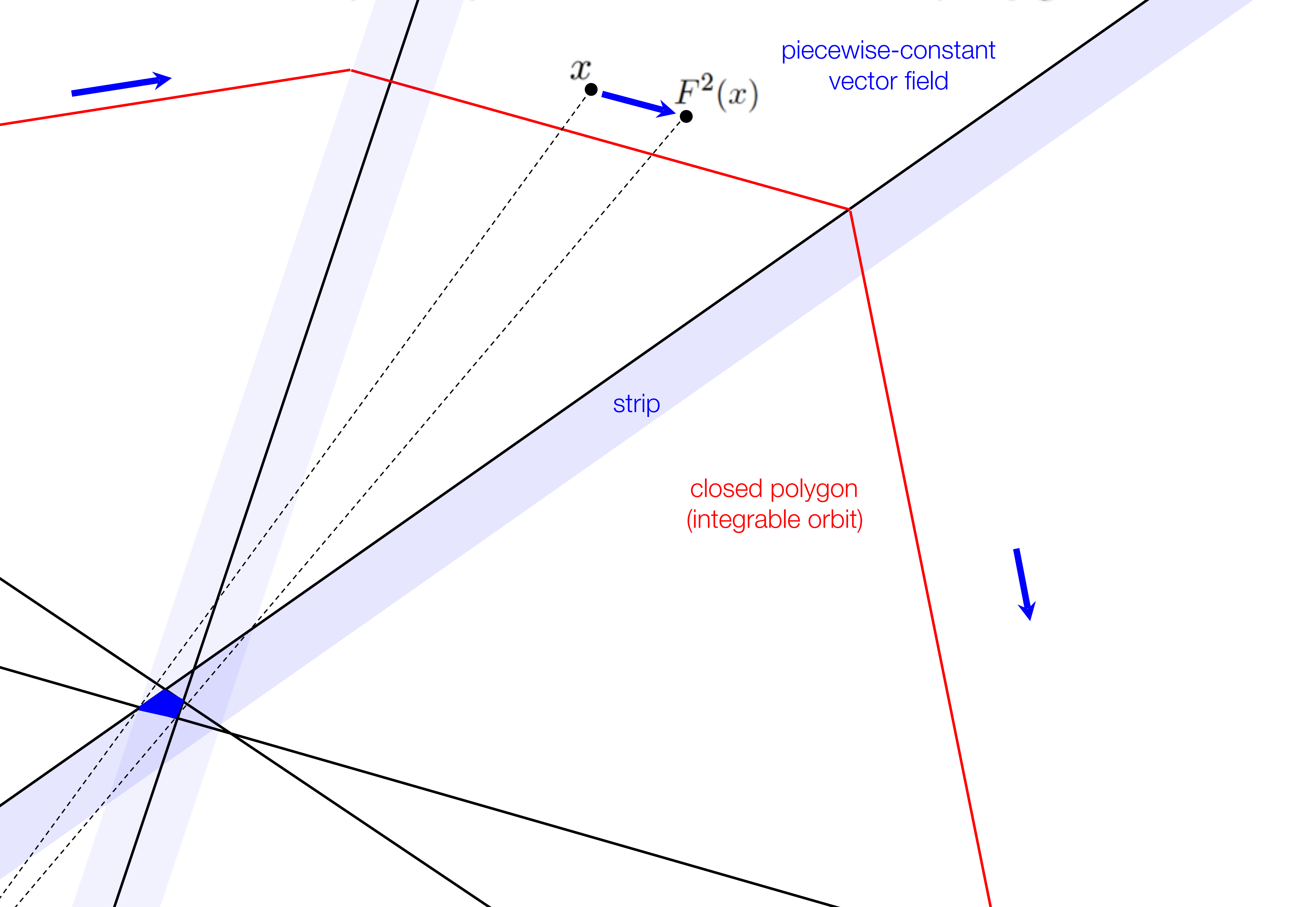
Linked strip maps: outer billiards of polygons



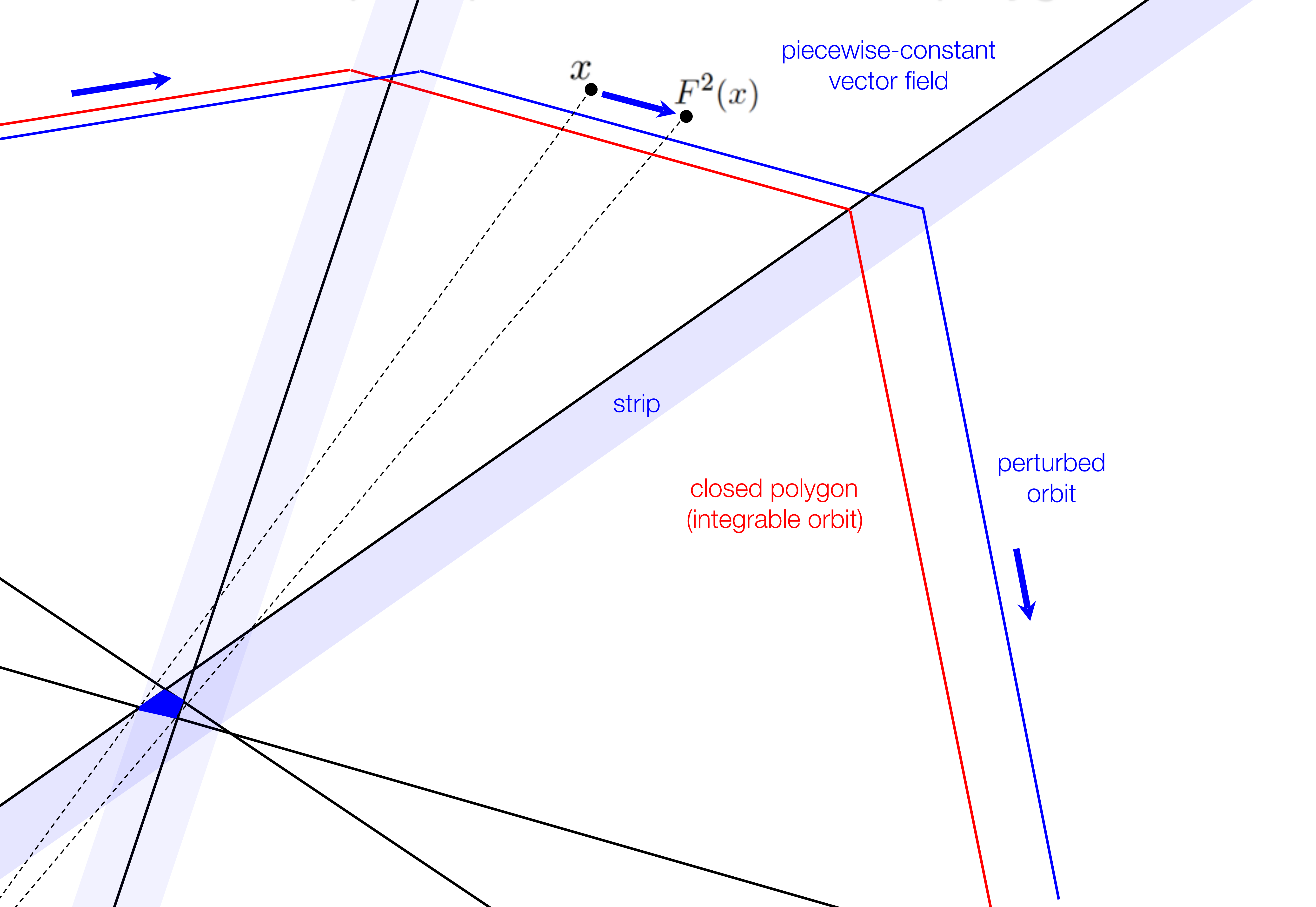
Linked strip maps: outer billiards of polygons



Linked strip maps: outer billiards of polygons



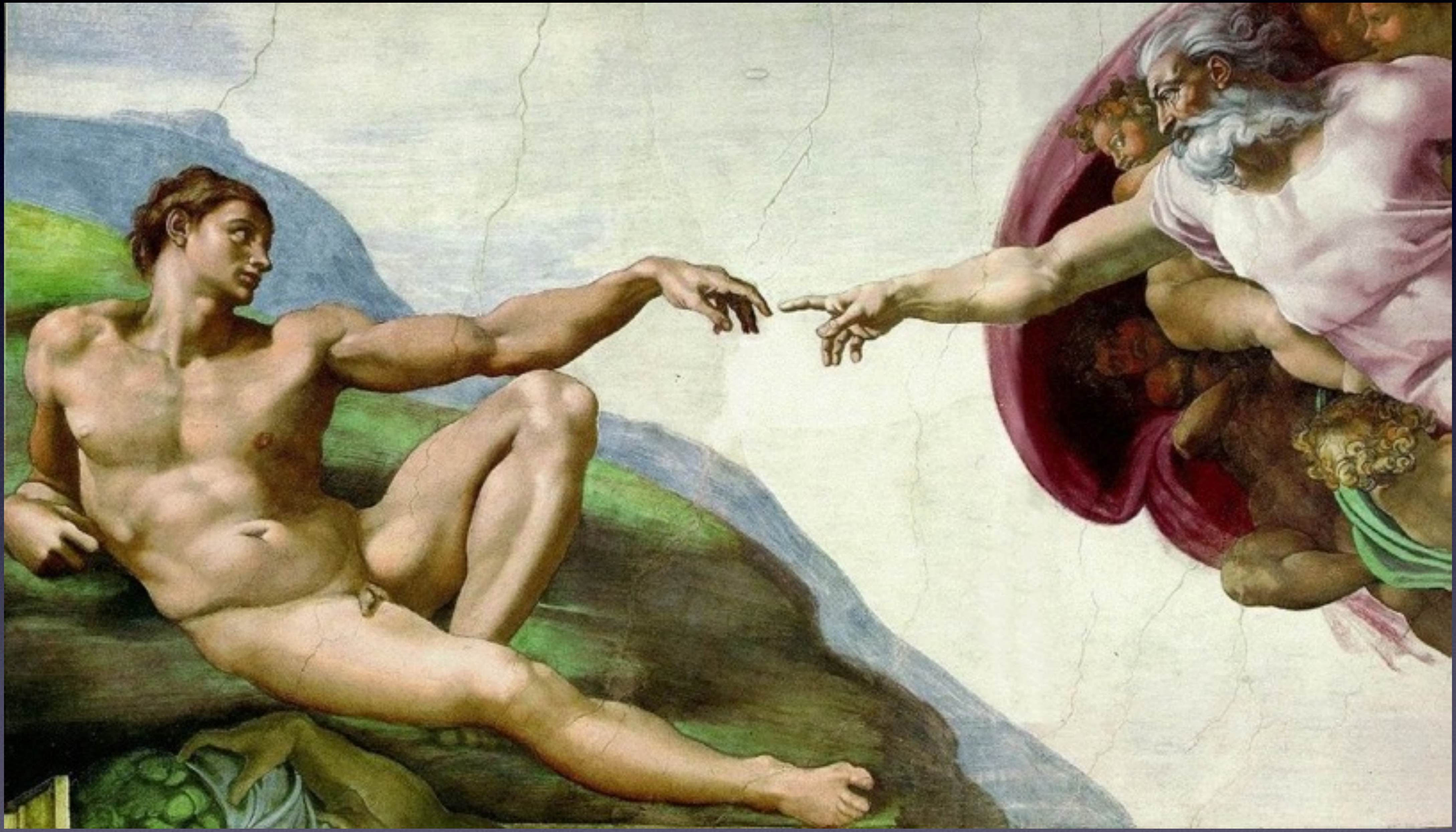
Linked strip maps: outer billiards of polygons



The arithmetic of chaos

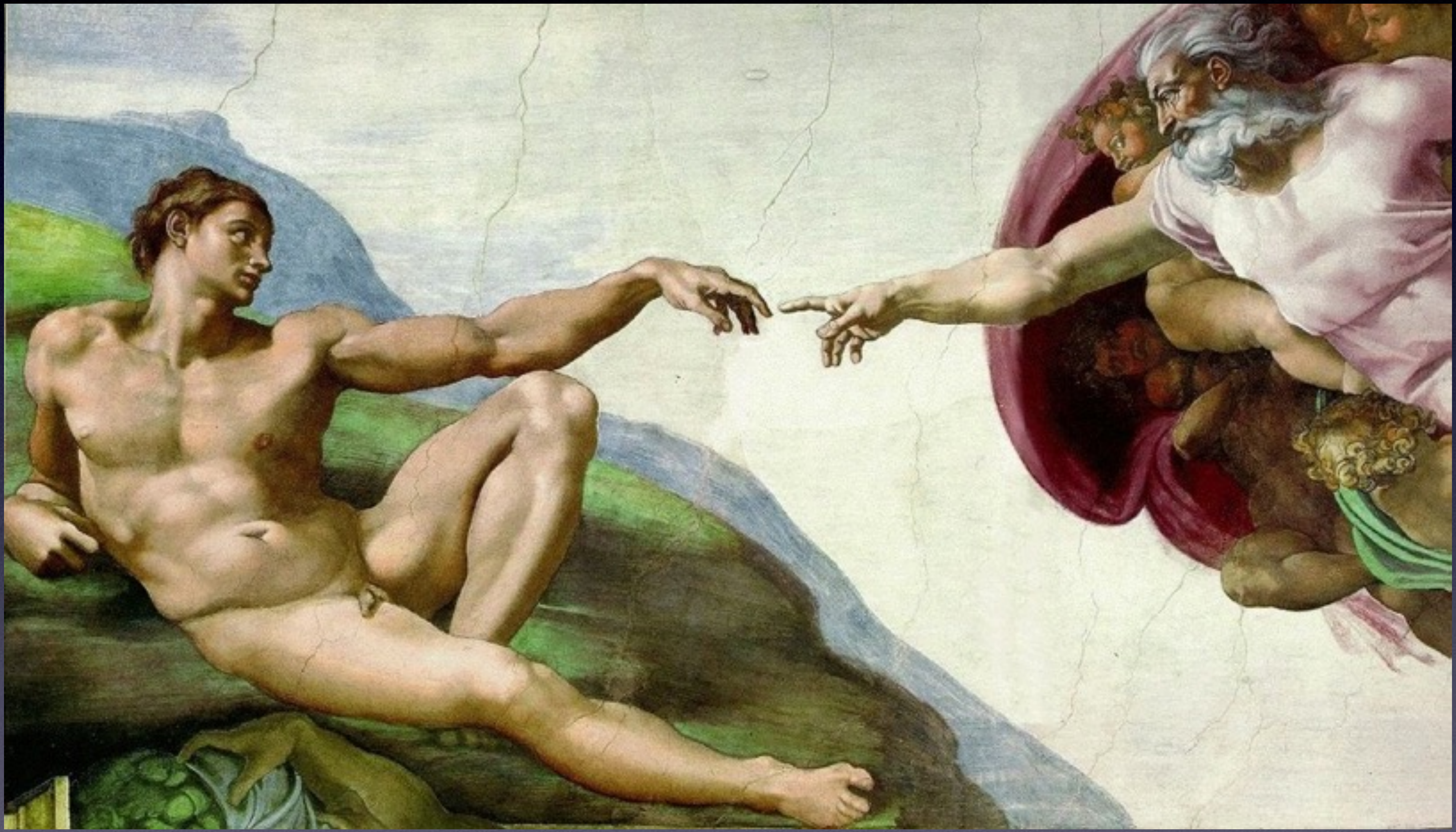
Franco Vivaldi
Queen Mary, University of London

God



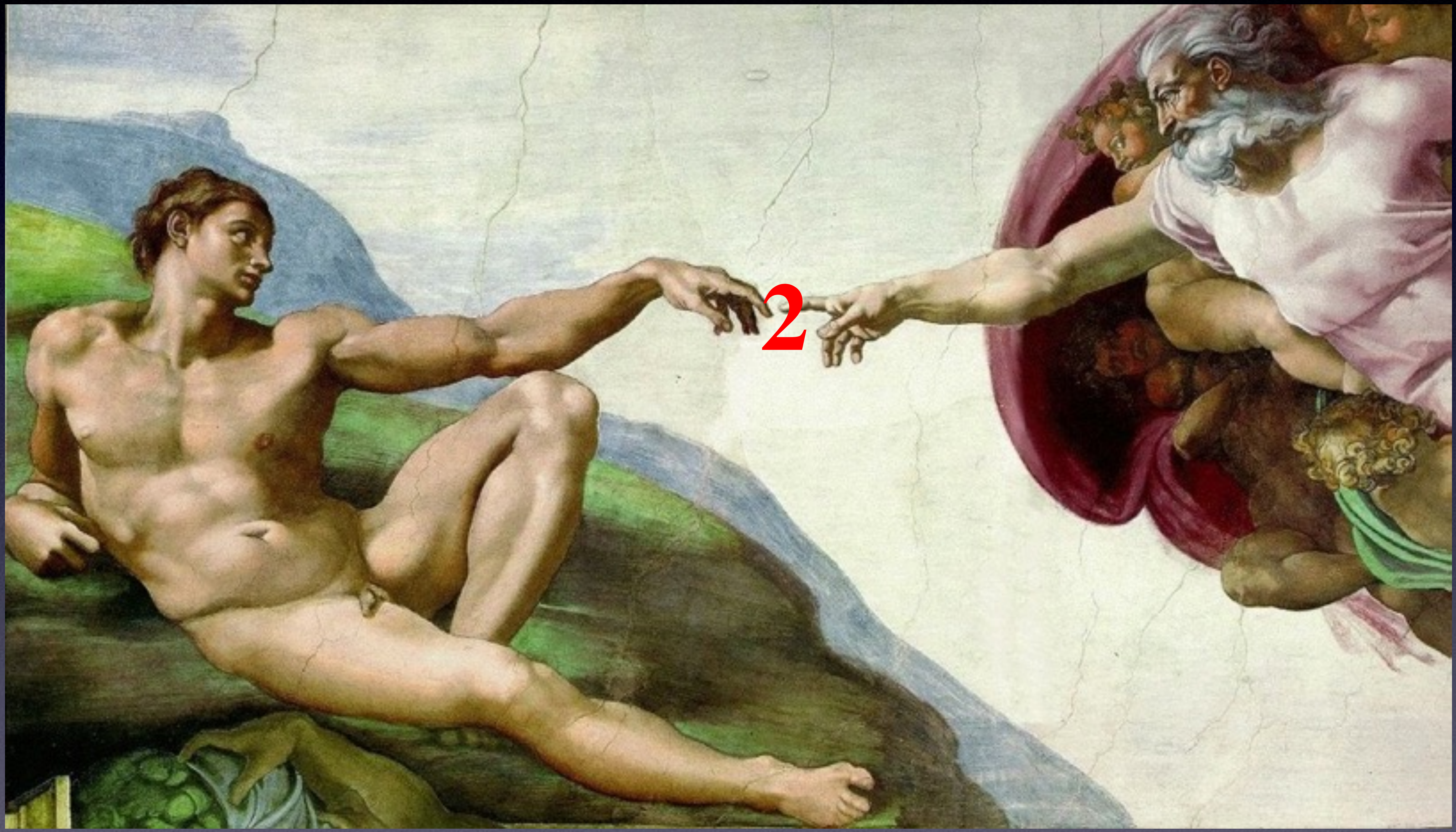
“God gave us the integers, the rest is the work of man”

L Kronecker

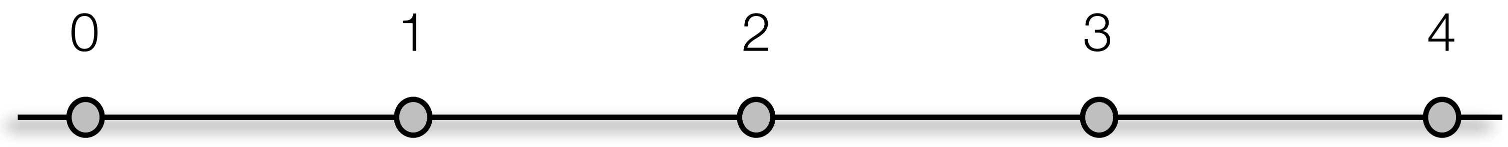


"God gave us the integers, the rest is the work of man"

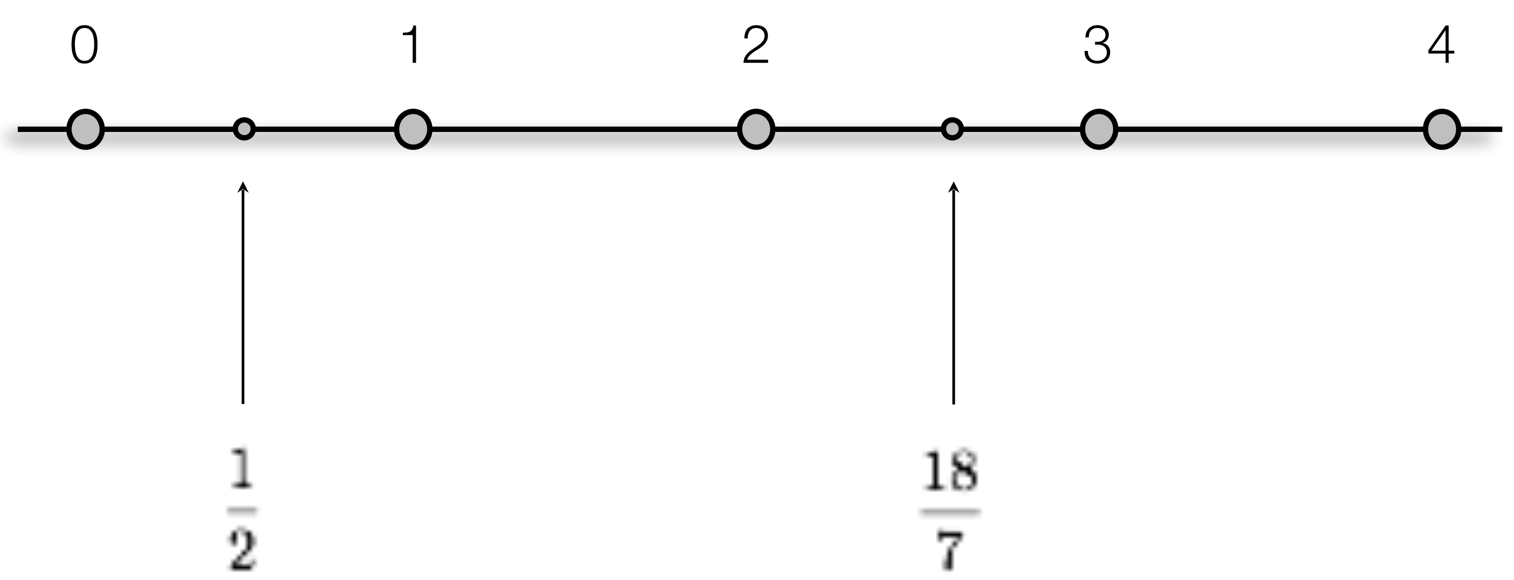
L Kronecker



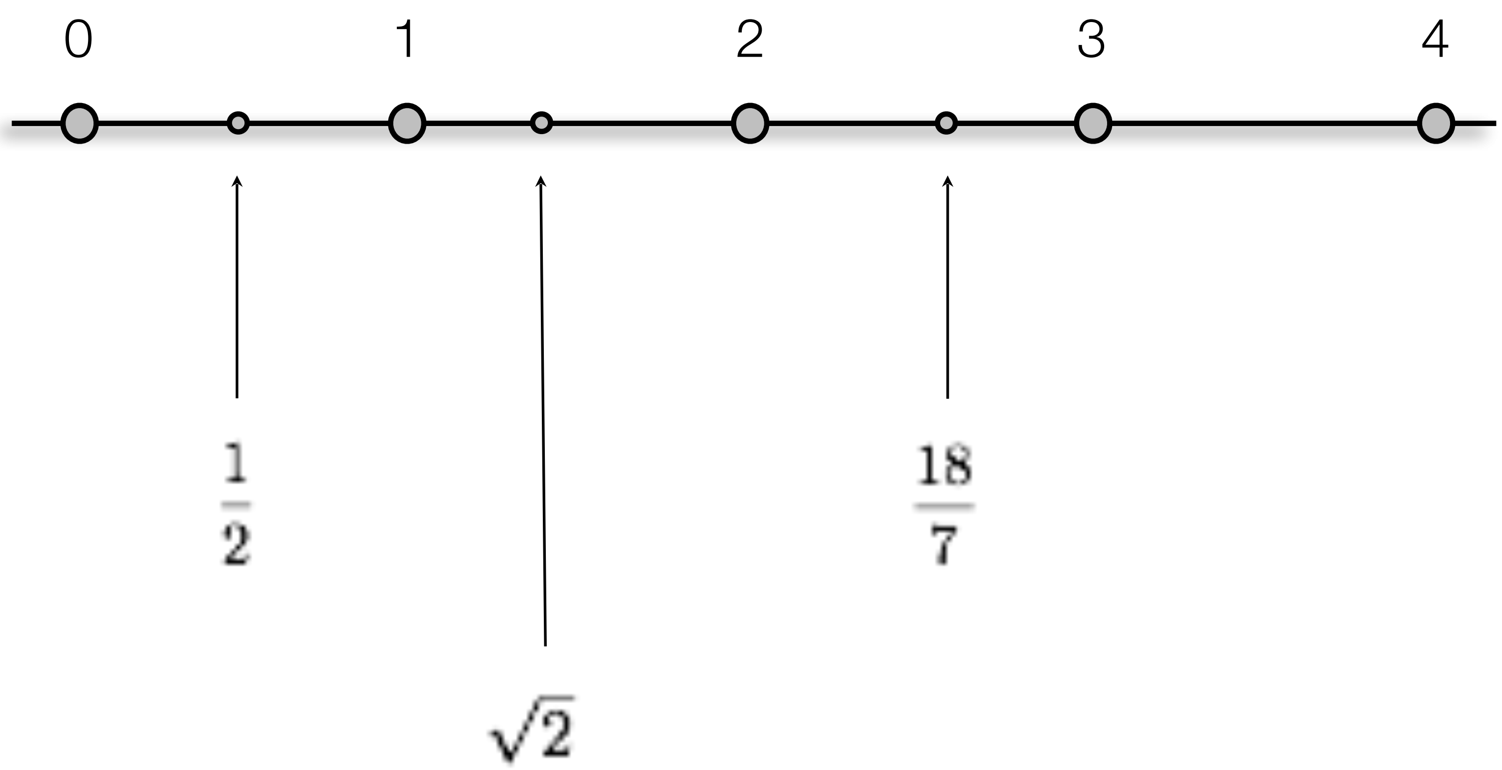
the work of man



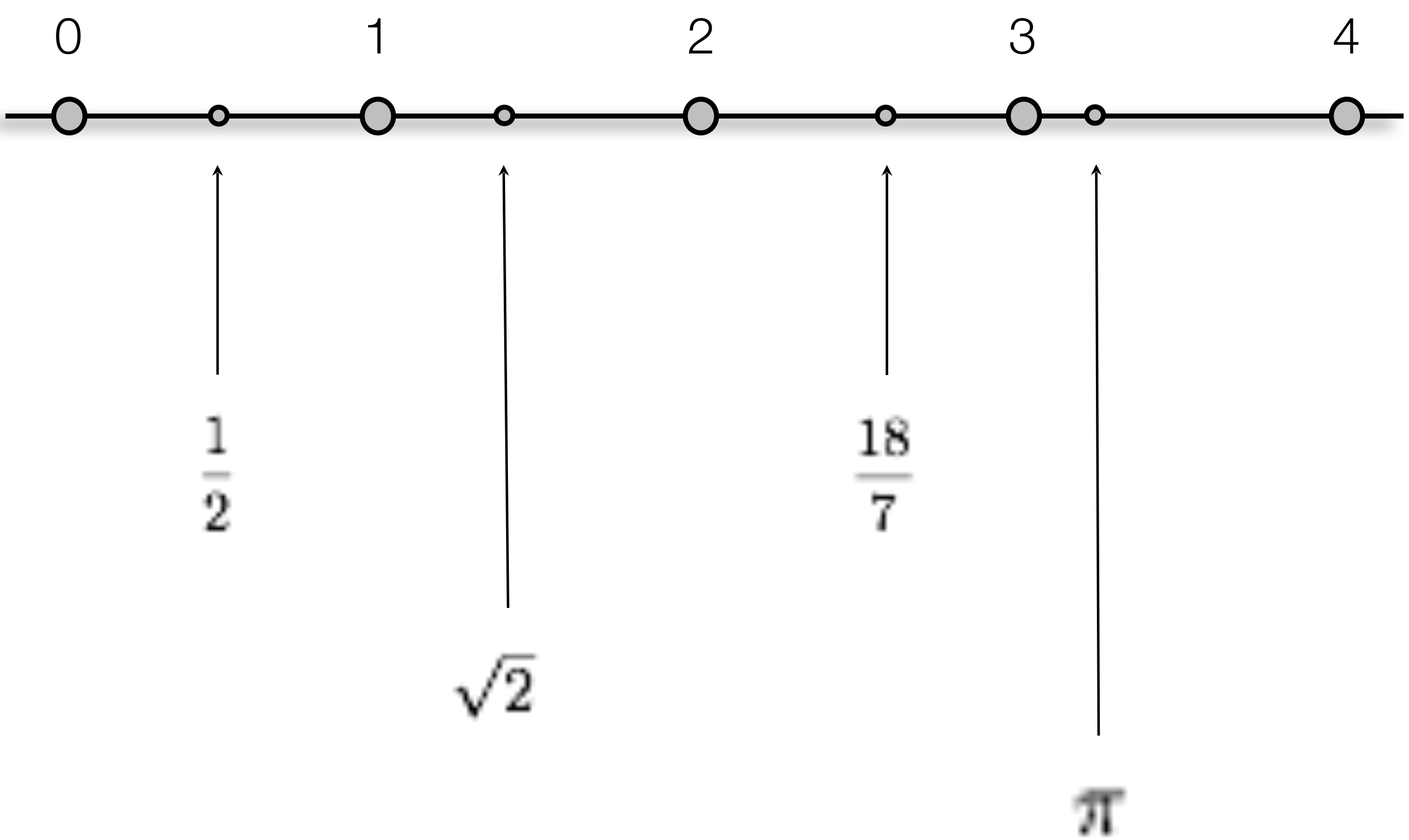
the work of man



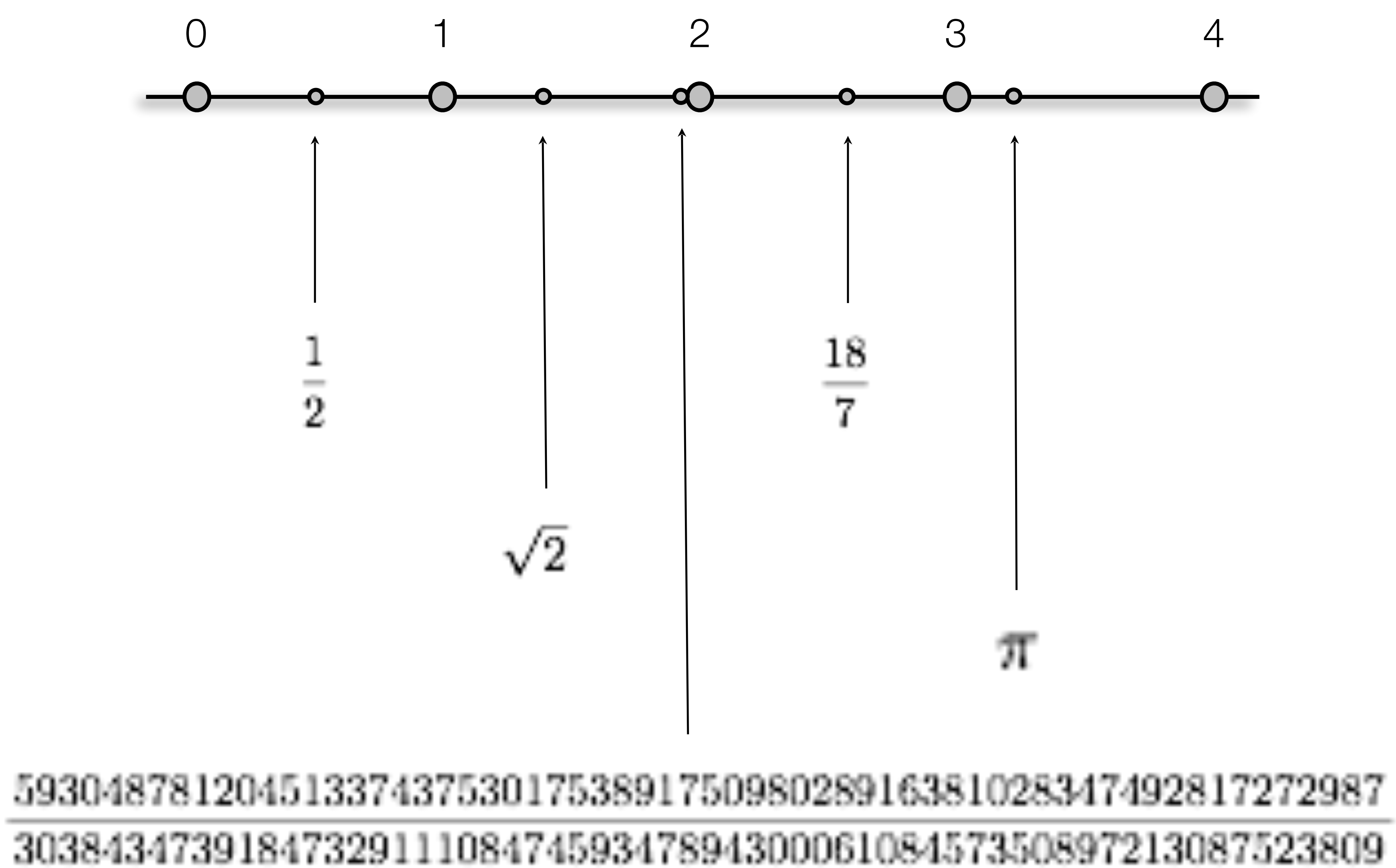
the work of man



the work of man



the work of man



computer programs to build numbers

print(123123123123123123123123123123123123123123123)

print(123) 15 times

computer programs to build numbers

print(123123123123123123123123123123123123123123123)

dumb

print(123) 15 times

smart

computer programs to build numbers

print(123123123123123123123123123123123123123123123)

dumb

print(123) 15 times

smart

print(3846264338327950288419716939937510582097494)

dumb

computer programs to build numbers

print(123123123123123123123123123123123123123123123)

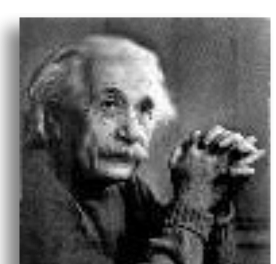
dumb

print(123) 15 times

smart

print(3846264338327950288419716939937510582097494)

dumb



smart

?

random numbers

A number is **random** if the shortest program that can build its digits is the **dumb** program

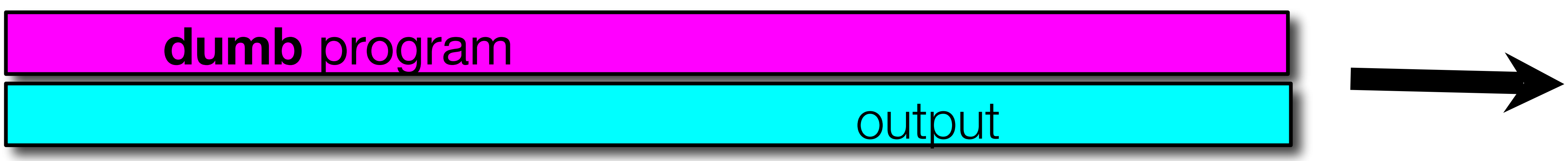
Kolmogorov

random numbers

A number is **random** if the shortest program that can build its digits is the **dumb** program

Kolmogorov

random



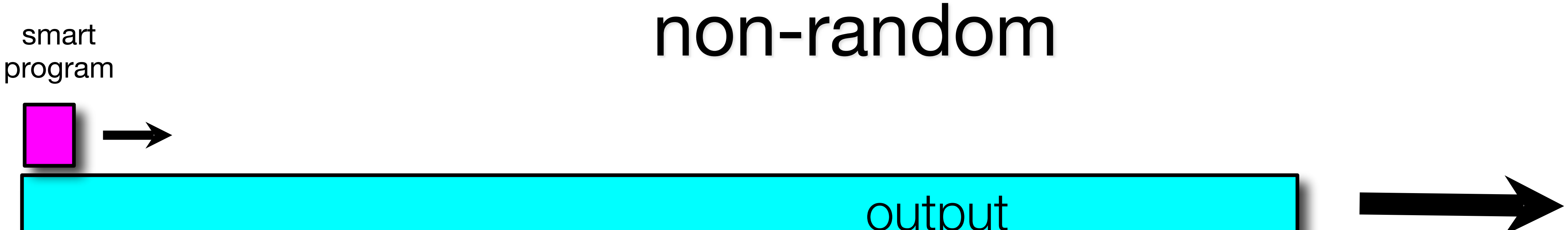
random numbers

A number is **random** if the **shortest** program that can build its digits is the **dumb** program


Kolmogorov

random

non-random



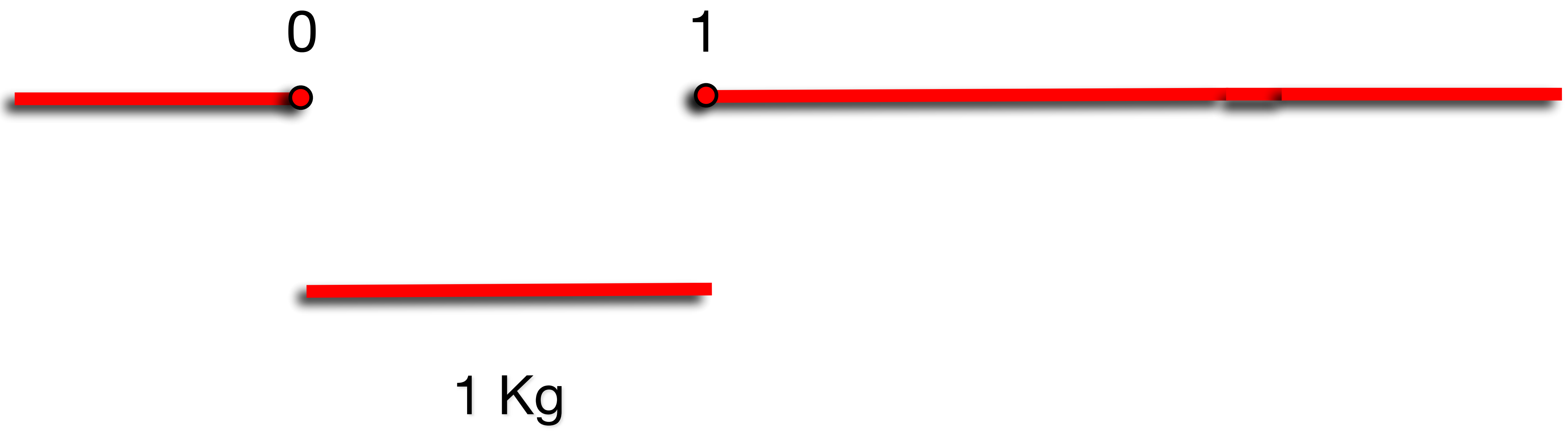
do numbers have mass?

do numbers have mass?

Lebesgue

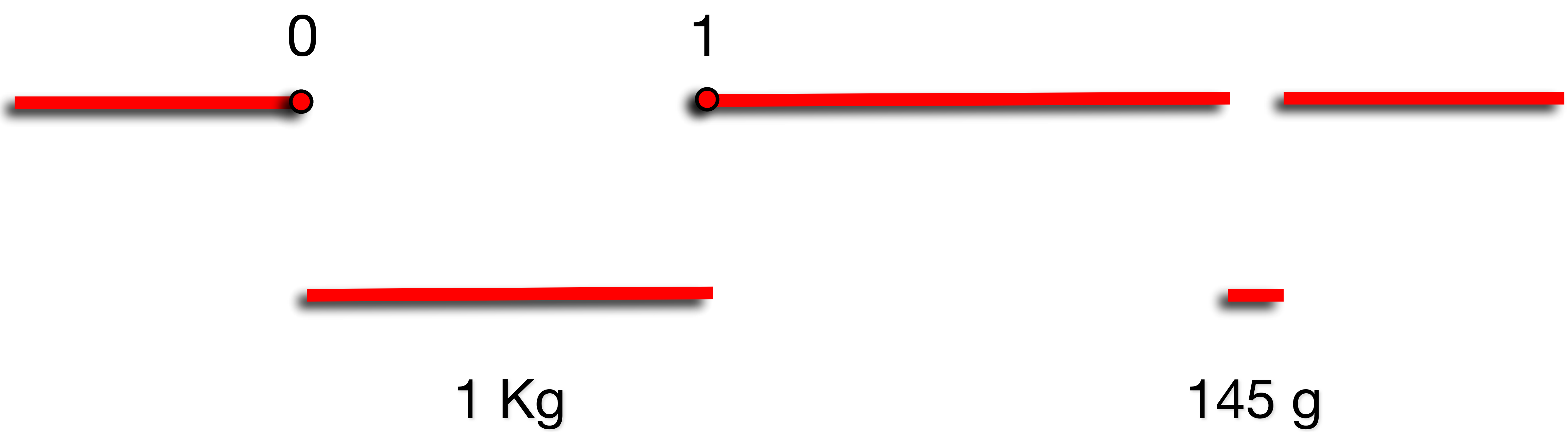
do numbers have mass?

Lebesgue



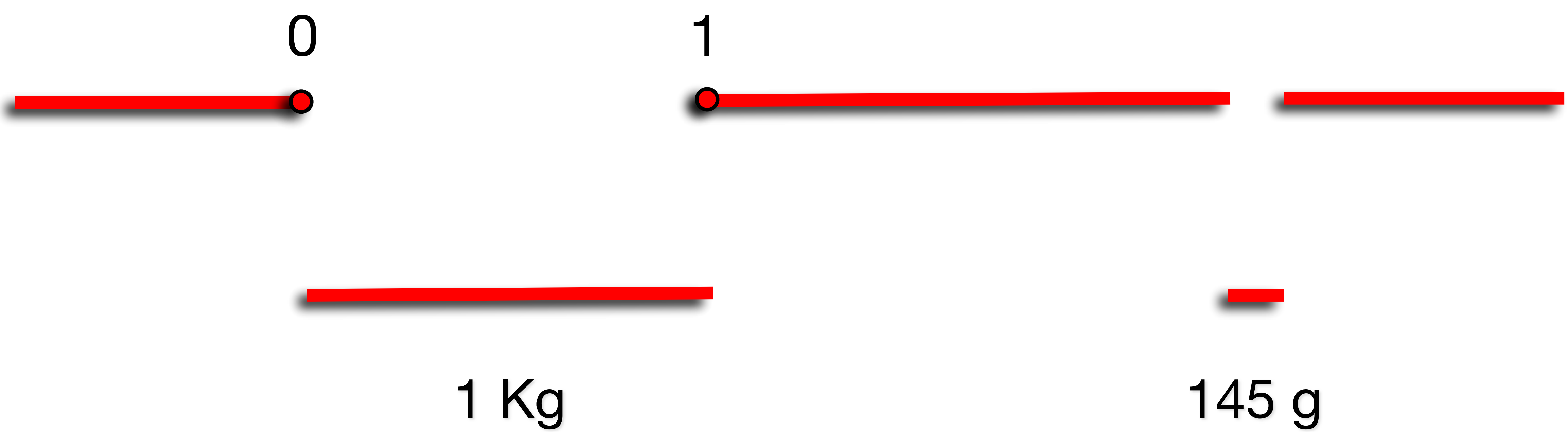
do numbers have mass?

Lebesgue



do numbers have mass?

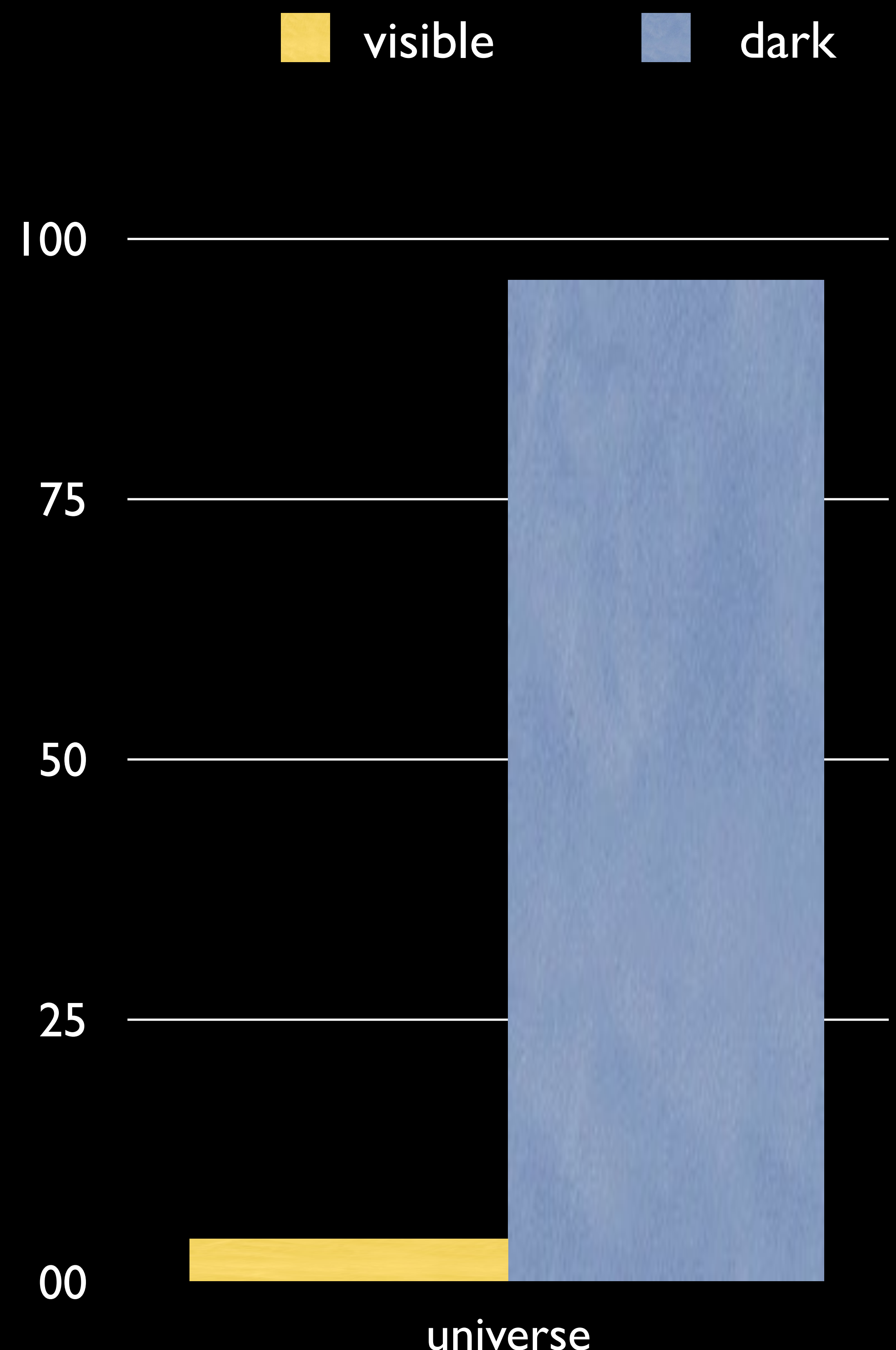
Lebesgue



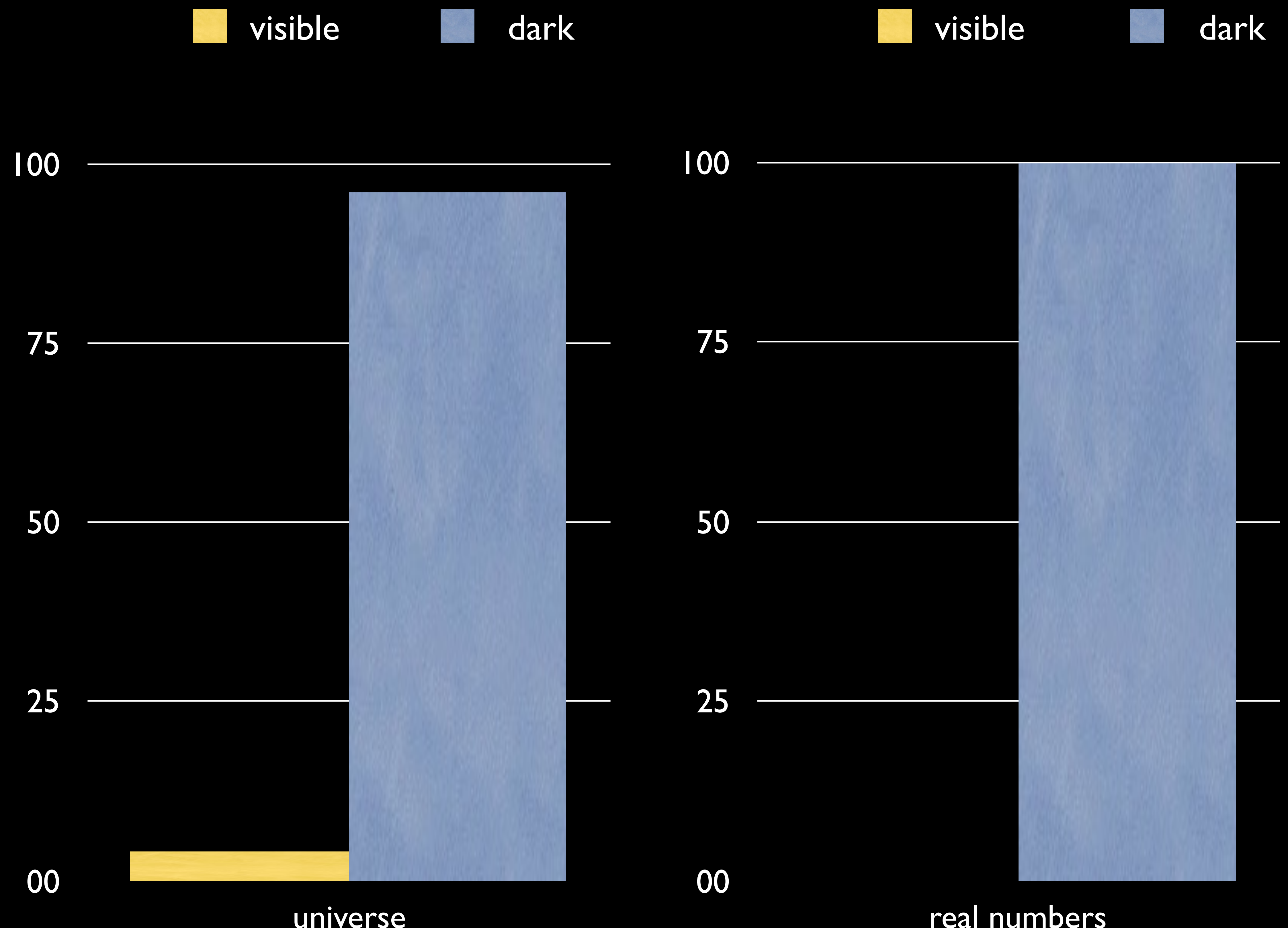
- The total mass of all fractions is <u>zero</u>

DARK MATTER

DARK MATTER



DARK MATTER



visible
dark
100
75
50
25
00
universe
visible
dark
100
75
50
25
00
real numbers

Theorem:

Theorem: the <u>random numbers</u> account for the <u>total mass</u> of the real number system.

Theorem: the random numbers account for the total mass of the real number system.

Random numbers are **dark**

Theorem: the random numbers account for the total mass of the real number system.

Random numbers are **dark**

- they can’t be defined individually, or talked about

Theorem: the random numbers account for the total mass of the real number system.

Random numbers are **dark**

- they can't be defined individually, or talked about
- they can't be computed, or stored in computers

Theorem: the random numbers account for the total mass of the real number system.

Random numbers are **dark**

- they can't be defined individually, or talked about
- they can't be computed, or stored in computers
- they can't be proved to be random

Theorem: the random numbers account for the total mass of the real number system.

Random numbers are **dark**

- they can't be defined individually, or talked about
- they can't be computed, or stored in computers
- they can't be proved to be random

Random numbers are not meant for humans.

Thank you for your attention