MAS/320 Number Theory: Coursework 2

Franco VIVALDI

http://www.maths.qmw.ac.uk/~fv/teaching/nt/nt.html

This coursework will be assessed and count towards your final mark for the course

DEADLINE: Wednesday of week 5, at 1:00 pm.

CONTENT: Infinite continued fractions.

M⋆icroESSAY: Write an essay on continued fractions. (Approximately 100 words, and no mathematical symbols.)

Problem 1. (Warm-ups on radicals.) Show that the value of each of the following expressions is an *integer*.

a) $$\left(7\frac{\sqrt{42}}{\sqrt{7}}+\frac{6\sqrt{18}}{\sqrt{3}}-2\sqrt{30}\sqrt{5}\right)\frac{2-\sqrt{6}}{\sqrt{54}-9}$$

b) $$\frac{(7-\sqrt{79})\,(3-\sqrt{10})^2\,(19+6\sqrt{10})}{(9-\sqrt{79})\,(8+\sqrt{79})}$$

c) $$\left(2+\sqrt{-1}\left(\frac{1+\sqrt{5}}{2}\right)\right)\left(2-\sqrt{-1}\left(\frac{1+\sqrt{5}}{2}\right)\right)\left(2+\sqrt{-1}\left(\frac{1-\sqrt{5}}{2}\right)\right)\left(2-\sqrt{-1}\left(\frac{1-\sqrt{5}}{2}\right)\right).$$

Problem 2. Find the simple continued fraction of each of the following quadratic surds

a) $\frac{4-\sqrt{7}}{2}$ *b)* $\sqrt{61}$ *c)* $\sqrt{\frac{17}{3}}$

Problem 3. Let n be a positive integer. Find the simple continued fraction of each of the following expressions

a) $\sqrt{n^2+1}$ *b)* $\sqrt{n^2+n}$ *c)* $\sqrt{n^{2k}+n},\quad k\geq 1.$

Problem 4. Determine the value of the following continued fractions

a) $[\overline{11}]$ *b)* $[3;3,\overline{1,5,2}]$ *c)* $[n;\overline{k,2n}],\quad k\geq 1.$

Problem 5.

(a) Let $x = [\,\overline{a; 2b}\,]$. Compute the continued fraction of $2x$ from that of x.

(b) Compute the continued fraction of nx, where

$$x = [a_0; na_1, a_2, na_3, \ldots, a_{2k}, na_{2k+1}, \ldots] \qquad n \geq 1.$$

Problem 6. In this exercise all estimates must be derived from the approximation theorem.

(a) Given that

$$\pi = [\,3; 7, 15, 1, 292, 1, \ldots],$$

find upper and lower bounds for the error involved in using C_1 and C_3 for π.

(b) Given that

$$e = [\,2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots]$$

show that in the interval $(\,e - \delta,\ e + \delta)$, with $\delta = 5 \cdot 10^{-8}$, there are no rationals with less than four decimal digits at denominator.

Problem 7. Let $\alpha = [0; a_1, a_2, \ldots]$ be a real number, chosen at random in the interval $(0, 1)$.

(a) What is the probability that $a_1 \geq m$?

(b) What is the probability that $a_1 = m$?

(c) Show that the probability that $a_2 \geq m$ is

$$\sum_{a=1}^{\infty} \frac{1}{a(m\,a + 1)}.$$

(d)* Use (c) to find the probability that $a_2 = 1$, *in closed form.*

[In each case, the numbers with the stated property form an interval, or a collection of disjoint intervals. The probability is defined as the total length of such intervals.]