MTH5117 Mathematical writing: Coursework 7

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DEADLINE: Sunday of week 10, at 23.55.

ASSESSED PROBLEMS [with allocated marks].
Problem 1: 4 [20]. Problem 2: [20].
Problem 3: [20]. Problem 4: [40].

Problem 1. Each of the following statements is (equivalent to) an implication, which may be true or false. For each implication [$\not\in$]

i) state the contrapositive;
ii) state the converse, and decide whether it is true or false;
iii) state the negation, and decide whether it is true or false.

[*First write each statement explicitly as an implication.*]

1. *Every rational number is a real number.*

2. *Every continuous real function is differentiable.*

3. *If a matrix is symmetric, then it equals its transpose.*

4. *The sum of two irrational numbers is irrational.*

5. *If a prime divides the product of two integers, then it divides one of them.*

Problem 2. Write the first few sentences of the proof of each statement, choosing an appropriate notation, structuring the beginning of the proof, and and identifying the RTP. [*See sections 7.3.1 for examples and 6.2 for notation. Understanding the meaning of the statements is immaterial.*]

1. *Any two antiderivatives of a generalised function differ only by a constant.*
2. *A subset of a metric space is open if and only if its complement is closed.*

Problem 3. Rewrite the given proof in good style. You must declare the strategy of the proof, set an appropriate notation, justify concisely every step, and provide appropriate clarifying remarks.

Prove that the curves $y = \sin(2x)$ *and* $y = -\sin(x/2)$ *intersect orthogonally at the origin.*

BAD PROOF:

$$f'(x) = \frac{d}{dx}\sin(2x) = 2\cos(2x)$$
$$g'(x) = \frac{d}{dx}(-\sin(x/2)) = -\frac{1}{2}\cos(x/2)$$
$$f'(0) = -1/g'(0); \quad f(0) = g(0). \qquad \square$$

[*What does it mean for two curves to intersect orthogonally at a point?*]

Problem 4.

Read carefully the text displayed on the next two pages. Then write a report on this document, comprising

i) a short title [$\notin$];

ii) two or three concise key points [$\notin$];

iii) a summary of the document [$\notin$, 150].

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The absolute value (or magnitude) of a real number x, denoted by $|x|$, is defined by the formula

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

The function $x \mapsto |x|$ is called the absolute value function; its graph is the line $y = x$ if $x \geq 0$, and the line $y = -x$ if $x < 0$.

Another way to obtain the absolute value of a real number is to square the number and then take the positive square root:

$$|x| = \sqrt{x^2}.$$

To solve an equation that contains absolute values, we write equivalent equations without absolute value, and then solve as usual. For instance, the equation $|2x - 3| = 7$ corresponds to two equations

$$\begin{array}{rcl} 2x - 3 &=& 7 \quad \text{if} \;\; 2x - 3 \geq 0, \;\; \text{or} \;\; x \geq 3/2; \\ -(2x - 3) &=& 7 \quad \text{if} \;\; 2x - 3 < 0, \;\; \text{or} \;\; x < 3/2. \end{array}$$

Hence the equation has two solutions: $x = 5$ and $x = -2$.

When we do arithmetic with absolute values, we can use the following rules, valid for all real numbers x, y:

(*a*) $|xy| = |x||y|$

(b) $|x/y| = |x|/|y| \qquad (y \neq 0)$

(c) $|x+y| \leq |x| + |y|$.

The inequality (c) becomes an equality precisely when x and y have the same sign.

From (a) we find

$$|-x| = |(-1)x| = |-1||x| = |x|.$$

It follows that, for all x and y, the numbers $|x-y|$ and $|y-x| = |-(x-y)|$ are the same. They give the distance between the points x and y on the real line.

The connection between absolute value and distance gives us an alternative way to define intervals. For example, the inequality $|x| < 5$ says that x lies between -5 and 5 on the real line

$$|x| < 5 \quad \Leftrightarrow \quad -5 < x < 5$$

and therefore such an inequality describes an open interval. More generally, if a is any real number and b is any positive real number, then the open (closed, respectively) interval with mid-point a and length $2b$ is written concisely as

$$|x-a| < b \qquad\qquad |x-a| \leq b$$

respectively.

Substituting $x-y$ for x and $y-z$ for y in the inequality (c), we obtain

$$|x-z| \leq |x-y| + |y-z|.$$

This is the so-called triangle inequality. It states that the direct path between x and z is not longer than the path from x to z which passes through a third point y. (If we think of x, y and z as points on the plane, they would be vertices of a triangle, hence the name.)