MTH5117 Mathematical writing: Coursework 6

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DEADLINE: Sunday of week 9, at 23.55.

ASSESSED PROBLEMS [with allocated marks].
Problem 1: [30]. Problem 2: 2, 4, 6 [15].
Problem 3: [15]. Problem 4: [40]

Problem 1. You are given two predicates, over a set X of people

$s(x, y) =$'x is a son of y' $\qquad$ $d(x, y) =$'x is a daughter of y'.

Write definitions of the following predicates, in some appropriate order so that the later definitions use only the given predicates and earlier definitions. (The order below is just alphabetical.)

$a(x, y) =$'*x is an aunt of y*'
$b(x, y) =$'*x is a brother of y*'
$c(x, y) =$'*x is a child of y*'
$F(x, y) =$'*x is the father of y*'
$f(x) =$'*x is female*'
$g(x, y) =$'*x is a grandchild of y*'
$h(x, y) =$'*x is a half-brother of y*'
$m(x) =$'*x is male*'
$M(x, y) =$'*x is the mother of y*'
$p(x, y) =$'*x is a parent of y*'
$t(x, y) =$'*x is a sister of y*'

Example:
x is a nephew of y $= \exists z \in X,\ s(x, z) \wedge (b(z, y) \vee t(z, y))$
So you can give this definition only after you have defined b (brother) and t (sister). The use of parentheses indicates that the operator OR is to be evaluated before the operator AND .

Problem 2. Write out the following sums in full. The sum ranges over all integers (positive and negative) satisfying the given conditions.

1. $\displaystyle\sum_{0\leqslant k-1\leqslant 3} a_k$
2. $\displaystyle\sum_{k^2+1<10} a_k$
3. $\displaystyle\sum_{k^2\leqslant k+2} a_{1-k}$
4. $\displaystyle\sum_{\substack{k\in 2\mathbb{Z}+1\\ |k|<5}} a_{k+1}$
5. $\displaystyle\sum_{\substack{|k-3|<5\\ \gcd(k,6)>1}} a_k$
6. $\displaystyle\sum_{k^2\leqslant 9} a_{k^2}$

[*Hint: Read section 3.2 of the web-book; item 6 is tricky.*]

Problem 3. Answer the question by giving explicit instructions, appropriate for an A-level student. You must set a suitable notation, and justify concisely every statement you make. Do not exceed half a page.

> *I have a function given by a cubic polynomial. How do I decide whether or not this function is odd?*

Problem 4. Read the text displayed on next two pages[1]. Then write a report on this document, comprising

– a short title [$\notin$, 5];
– two concise key points, highlighting some essential concepts [$\notin$, 10];
– a summary of the document [$\notin$, 150].

[1] Main source: A F Beardon, *Algebra and Geometry* Cambridge University Press, Cambridge (2005).

Let $z = x + iy$ be a complex number. The complex conjugate $\overline{z}$ of z is given by $\overline{z} = x - iy$. Geometrically, z and $\overline{z}$ are mirror images of each other in the real axis. From the definition, it immediately follows that $\overline{\overline{z}} = z$.

Several quantities involving complex numbers may be defined using complex conjugation. We find

$$\frac{z + \overline{z}}{2} = x = \mathrm{Re}(z) \qquad \frac{z - \overline{z}}{2i} = y = \mathrm{Im}(z).$$

Thus z is a real number (a complex number with zero imaginary part) precisely when $\overline{z} = z$. Furthermore, we have

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2$$

which shows that $z\overline{z}$ is real and non-negative. We also see that the equation $z\overline{z} = r^2$ defines a circle of radius r on the complex plane. The modulus $|z|$ of z is defined by

$$|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}, \tag{1}$$

where we take the non-negative root of the real number $x^2 + y^2$. Note that if z is a real number ($y = 0$), then the complex absolute value defined in equation (1) coincides with the real absolute value $|x| = \sqrt{x^2}$.

For any complex numbers z, w the following holds

$$\overline{z + w} = \overline{z} + \overline{w}, \qquad \overline{z - w} = \overline{z} - \overline{w}, \qquad \overline{zw} = \bar{z}\bar{w}. \tag{2}$$

Indeed, let $w = r + is$, with $r, s \in \mathbb{R}$. Since $z + w = (x + r) + i(y + s)$, we find

$$\overline{z + w} = (x + r) - i(y + s) = x - iy + r - is = \overline{z} + \overline{w}.$$

We leave it to the reader to verify the remaining identities in (2).

If $z \neq 0$, and $w = 1/z$, then $\overline{zw} = \bar{z}\,\bar{w} = \overline{1} = 1$, so that $\overline{w} = 1/\overline{z}$. More generally, if $w \neq 0$, then

$$\overline{z/w} = \overline{z}/\overline{w}.$$

Combining this result with (2), we conclude that complex conjugation preserves all four arithmetical operations. This result may be expressed synthetically as follows: for any complex numbers z, w we have

$$\overline{z \odot w} = \overline{z} \odot \overline{w}$$

where $\odot$ denotes any of the four arithmetical operators. (Does the above formula continue to hold if $\odot$ denotes exponentiation?)

Next, for any complex numbers z and w, with $w \neq 0$, we can always write z/w as $z\overline{w}/w\overline{w}$, where the denominator $w\overline{w}$ of the second quotient is real and positive. With z and w as above, we find

$$\frac{z}{w} = \frac{x+iy}{r+is} = \frac{(x+iy)(r-is)}{r^2+s^2} = \frac{xr+ys}{r^2+s^2} - i\frac{xs+yr}{r^2+s^2}.$$

Now, the quantities

$$\frac{xr+ys}{r^2+s^2} \qquad\qquad -\frac{xs+yr}{r^2+s^2}$$

are real numbers which represent, respectively, the real and imaginary parts of z/w.

Taking $z = w$ in the third identity in (2), we see that $\overline{z^2} = \bar{z}\bar{z} = (\overline{z})^2$. Likewise, $\overline{z^3} = \overline{z^2 z} = \overline{z^2}\,\overline{z} = (\overline{z})^2\overline{z} = (\overline{z})^3$. Continuing this process by induction, we find that for all positive integers n

$$\overline{z^n} = (\overline{z})^n.$$

Consider now a polynomial in z, with complex coefficients

$$p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0. \tag{3}$$

Applying the above rules, we obtain

$$\begin{aligned}
\overline{p(z)} &= \overline{a^n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0} \\
&= \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a}_0 \\
&= \overline{a}_n\overline{(z^n)} + \overline{a}_{n-1}\overline{(z^{n-1})} + \cdots + \overline{a}_1\overline{z} + \overline{a}_0 \\
&= \overline{a}_n\overline{z}^n + \overline{a}_{n-1}\overline{z}^{n-1} + \cdots + \overline{a}_1\overline{z} + \overline{a}_0.
\end{aligned}$$

Thus the conjugate of a polynomial in z is a polynomial in $\overline{z}$, with conjugate coefficients. By the same token, the conjugate of a rational function in z (a ratio $p(x)/q(z)$, with p and q complex polynomials) is a rational function in $\overline{z}$. In particular, for any algebraic equation $r(z) = 0$ there is a twin algebraic equation $\overline{r(z)} = 0$ involving conjugate quantities.