MTH5117 Mathematical writing: Coursework 5

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DEADLINE: Sunday of week 8, at 23.55.

ASSESSED PROBLEMS [with allocated marks].
Problem 1: 1, 2 [10]. Problem 2: 4, 7 [30].
Problem 3: 1, 3 [20]. Problem 5: [40].

Problem 1. Write with symbols, using at least one quantifier.

1. The reciprocal of an integer may still be an integer.

2. The equations $f(x) = 0$ and $g(x) = 0$ have no real solutions in common.

Problem 2. Consider the following implications (A, B are sets, f is a real function).

1. If $A \subset B$, then $A \setminus B$ is the empty set.
2. If $x \in (A \setminus B)'$, then $x \notin B$.
3. If $(x, x) \in A \times A$, then $x \in A$.
4. If f is even, then $|f|$ is also even.
5. If f is decreasing, then $-f$ is increasing.
6. If $|f|$ is increasing, then f is monotonic.
7. If n and m are coprime, then mn is not a square.

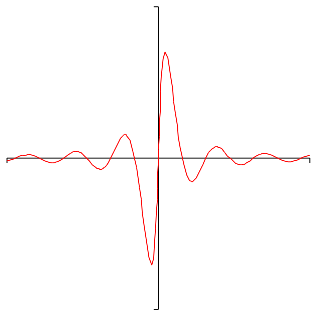
Of each implication:

(a) state the converse, and decide whether it's true or false.
(b) state the contrapositive, and decide whether it's true or false.

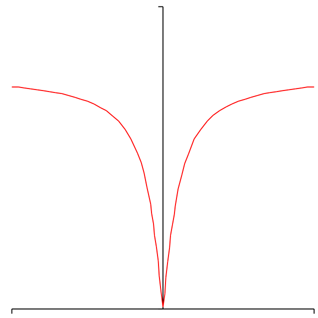
[*You may use symbols.*]

Problem 3. Describe the behaviour of the following functions. [$\notin$, 30].

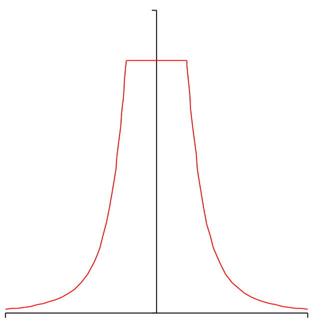
1)



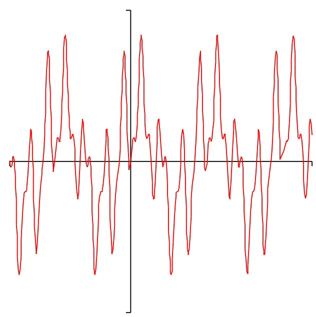
2)



3)



4)



Problem 4. Let $\mathbf{P}(X)$ denote the power set of a set X. What is $\mathbf{P}(\mathbf{P}(\mathbf{P}(\emptyset)))$? Explain in detail.

Problem 5. Read the text displayed on the next two pages[1]. Then write a report on this document, comprising

- a short title [$\notin$, 5];
- two/three very concise key points, highlighting the most important concepts [$\notin$, 10];
- a summary of the document [$\notin$, 150].

[1]Source: R L Finney and G B Thomas, *Calculus*, Addison-Wesley, Reading MA (1990).

The function $y = \sin(x)$ is not one-to-one: it runs through its full range of values from -1 to 1 twice on every interval of length 2π. However, if we restrict the domain of the sine to the interval from $-\pi/2$ to $\pi/2$, we find that the restricted function

$$y = \sin(x), \qquad -\pi/2 \leq x \leq \pi/2$$

is one-to-one. It therefore has an inverse, which we denote as

$$y = \sin^{-1}(x) \qquad \text{or} \qquad y = \arcsin(x).$$

Thus if $x = \sin(y)$, then y is the arc on the unit circle whose sine is x. For every value of x in the interval $[-1, 1]$, $y = \sin^{-1}(x)$ is the number in the interval $[-\pi/2, \pi/2]$ whose sine is x.

The superscript -1 in $\sin^{-1}(x)$ is not an exponent: it means 'inverse', rather than 'reciprocal'. The reciprocal of $\sin(x)$ is

$$\sin(x)^{-1} = \frac{1}{\sin(x)} = \csc(x).$$

The function $\sin^{-1}$ is odd, meaning that

$$\sin^{-1}(-x) = -\sin^{-1}(x)$$

which is valid for every x in the domain of the arc sine.

Like the sine function, the cosine function $y = \cos(x)$ is not one-to-one, but its restriction to the interval $[0, \pi]$ is one-to-one. The restricted function therefore has an inverse,

$$y = \cos^{-1}(x),$$

which we call the arc cosine of x. It is possible to show that

$$\cos^{-1}(x) + \cos^{-1}(-x) = \pi$$

and that

$$\sin^{-1}(x) + \cos^{-1}(-x) = \pi/2.$$

The other four basic trigonometric functions, tangent, secant, cosecant, and cotangent, also have inverses, when suitably restricted. The inverse of

$$\tan(x) \qquad -\pi/2 \leq 0 \leq \pi/2$$

is denoted by

$$\tan^{-1}(x).$$

The domain of the arc-tangent is the entire real line, and the co-domain is the open interval $(-\pi/2, \pi/2)$. Like the arc sine, the arc tangent is an odd function

$$\tan^{-1}(x) = -\tan^{-1}(-x).$$

The inverses of the restricted functions

$$\begin{array}{ll} y = \cot(x) & 0 < x < \pi \\ y = \sec(x) & 0 \le x \le \pi, \quad x \ne \pi/2 \\ y = \csc(x) & -\pi/2 \le x \le \pi/2, \quad x \ne 0 \end{array}$$

are chosen in such a way as to satisfy the relationships

$$\begin{aligned} \cot^{-1}(x) &= \pi/2 - \tan^{-1}(x) \\ \sec^{-1}(x) &= \cos^{-1}(1/x) \\ \csc^{-1}(x) &= \sin^{-1}(1/x). \end{aligned} \tag{1}$$

We remark that the above choices are not unique. For instance, some authors choose $\sec^{-1}(x)$ to lie between 0 and $\pi/2$, when x is positive, and between $-\pi$ and $-\pi/2$ when x is negative. This has the advantage of simplifying the formula for the derivative of $\sec^{-1}$, but the disadvantage of failing to satisfy the identity (1) when x is negative.