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Dynamical Systems (LTCC)
Problem Sheet 5

Problem 5 ("Piecewise linear Markov maps")

A one-dimensional Markov map $f : I \to I$ with Markov partition $\{I_0, I_1, \ldots, I_{N-1}\}$ is said to be piecewise linear if the slope is constant on each part, i.e., $f'(x) = \gamma_\ell$ for $x \in \mathrm{int}(I_\ell)$. If $\chi_\ell(x)$ denotes the characteristic function of I_ℓ (i.e. $\chi_\ell(x) = 1$ if $x \in I_\ell$, $\chi_\ell(x) = 0$ if $x \notin I_\ell$) then

$$\rho(x) = \sum_{\ell=0}^{N-1} \rho_\ell \chi_\ell(x) \qquad (1)$$

is a solution of the Frobenius-Perron equation for suitable values of ρ_ℓ ("the invariant density is piecewise constant").

a) Show that equation (1) solves the Frobenius-Perron equation if $(\rho_0, \rho_1, \ldots, \rho_{N-1})$ is the (right-)eigenvector of the so called transfer matrix

$$T_{kj} = \frac{A_{jk}}{|\gamma_j|}$$

with eigenvalue one.

b) Consider the map $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 3x + 1/4 & \text{if} \quad 0 \le x \le 1/4 \\ -2x + 3/2 & \text{if} \quad 1/4 < x < 3/4 \\ 2x - 3/2 & \text{if} \quad 3/4 \le x \le 1 \end{cases} .$$

Sketch the graph of the map, determine a Markov partition, and state the corresponding topological transition matrix. Determine the transfer matrix of the map f and compute the invariant density. Compute the Lyapunov exponent of the map (you may assume the density to be ergodic).