Wolfram Just
(Queen Mary / University of London)

Dynamical Systems (LTCC)
Problem Sheet 4

Problem 4 ("Symbolic dynamics")

Consider the tent map $T : [0,1] \to [0,1]$, $T(x) = 1 - |2x-1|$ with Markov partition $I_0 = [0,1/2]$, $I_1 = [1/2,1]$. For each (admissible) symbol sequence $.\sigma_0\sigma_1\sigma_2\ldots$ define the corresponding "Milnor-Thurston sequence" $.\tau_0\tau_1\tau_2\ldots$ by

$$\begin{aligned} \tau_0 &= \sigma_0 \\ \tau_k &= \begin{cases} \sigma_k & \text{if the string } \sigma_0\sigma_1\ldots\sigma_{k-1} \text{ contains an even number of ones} \\ 1-\sigma_k & \text{if the string } \sigma_0\sigma_1\ldots\sigma_{k-1} \text{ contains an odd number of ones} \end{cases} \end{aligned}$$

and use this sequence as a binary code for a number in [0,1]

$$x = \sum_{k=0}^{\infty} \frac{\tau_k}{2^{k+1}} .$$

The mapping $h : .\sigma_0\sigma_1\sigma_2\ldots \mapsto x$ defined in this way (semi-)conjugates the symbol shift to the tent map

$$\begin{array}{llcl} T: & x & \longmapsto & T(x) \\ & h\uparrow & & \uparrow h \\ S: & .\sigma_0\sigma_1\sigma_2\ldots & \longmapsto & .\sigma_1\sigma_2\sigma_3\ldots \end{array}$$

a) Show that the previous statement is correct, i.e., show that $x = h(.\sigma_0\sigma_1\sigma_2\ldots)$ and $y = h(.\sigma_1\sigma_2\sigma_3\ldots)$ obey $y = T(x)$.

b) Show that the map h is onto [0,1] (surjectivity).

c) Characterise the subset in $[0,1]$ where h fails to be injective, i.e., characterise the set

$$J := \{x \in [0,1] : h^{-1}(x) \text{ contains more than a single point } \}$$

in terms of the tent map T.