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Dynamical Systems (LTCC)
Problem Sheet 3

Problem 3 ("Adiabatic elimination")

Centre manifold techniques can be used to explore the behaviour in an entire neighbourhood of a bifurcation point in parameter space. For that purpose consider the driven van der Pol equation

$$\begin{aligned}\dot{u}(t) &= u(t) - \sigma v(t) - u(t)(u^2(t) + v^2(t)) \\ \dot{v}(t) &= \sigma u(t) + v(t) - v(t)(u^2(t) + v^2(t)) - \gamma \end{aligned} \tag{1}$$

a) For $\gamma = \gamma_* = 3\sqrt{2}/8$ and $\sigma = \sigma_* = \sqrt{5}/4$ the system has the fixed point $(u_*, v_*) = (\sqrt{10}/4, \sqrt{2}/4)$. Show that the fixed point is non-hyperbolic and compute the centre eigenspace (i.e. the eigenvector for vanishing eigenvalue).

b) Consider a (small) interval of parameter values, $\gamma = \gamma_* + a(\delta\gamma)^2$, $\sigma = \sigma_* = 1/2$ where a takes either the value $a = 1$ or $a = -1$, and rewrite the equations (1) formally as three dimensional system

$$\delta\dot{u} = f(\delta u, \delta v, \delta\gamma), \quad \dot{v} = g(\delta u, \delta v, \delta\gamma), \quad \delta\dot{\gamma} = 0 \tag{2}$$

where $\delta u = u - u_*$, $\delta v = v - v_*$. Compute the right hand sides in equation (2) up to (and including) quadratic terms in $\delta\gamma$, δu, δv.

c) Determine the centre eigenspace of the system (2) at $(\delta u, \delta v, \delta\gamma) = (0, 0, 0)$ and write down the centre manifold

$$\delta u = h(\delta v, \delta\gamma)$$

to first order in the arguments (i.e. just the linear part).

d) Write down the equation of motion on the centre manifold

$$\delta\dot{v} = f(h(\delta v, \delta\gamma), \delta v, \delta\gamma)$$

up to (and including) second order in the arguments. Use an additional linear transformation to reduce the equation of motion to normal form. For which values of $a(\delta\gamma)^2$ (i.e. on which side of the bifurcation point/line) is the pair of fixed points generated?