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Queen Mary University of London

Dynamical Systems (LTCC)
Problem Sheet 2

Problem 2 ("Conjugacy")

The phase portrait of a stable focus and a stable node may "look" differently, but a closer inspection will show that both are conjugate. For that purpose consider the two systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} .$$

Obviously $(x_*, y_*) = (0, 0)$ is s stable focus and $(\xi_*, \eta_*) = (0, 0)$ is a stable node (cf. the eigenvalues of the two matrices).

a) To establish a homeomorphism rewrite both equations in polar coordinates, $x = r\cos(\varphi)$, $y = r\sin(\varphi)$ and $\xi = \rho\cos(\psi)$, $\eta = \rho\sin(\psi)$ respectively, and compute the time dependent solutions $r(t)$, $\varphi(t)$ and $\rho(t)$, $\psi(t)$. Eliminate the time t so that you end up with a mapping $(r, \varphi) \mapsto (\rho, \psi)$ (with appropriate choices for initial conditions).

b) Rewrite this mapping in terms of Cartesian coordinates $(x, y) \mapsto (\xi, \eta)$ and show that the mapping is invertible and continuous. Is the mapping smooth? What is the "geometric meaning" of this transformation?