Problem 5

a) As we deal with densities of measures we do not have to worry about sets of Lebesgue measure zero, i.e., about boundary points of the partition.

$$\begin{aligned}
\sum_{y\in f^{-1}(x)} \frac{1}{|f'(y)|}\rho(y) &= \sum_k \sum_{y\in f^{-1}(x)} \frac{1}{|f'(y)|}\chi_k(y)\rho_k \\
&= \sum_k \frac{1}{|\gamma_k|}\rho_k \sum_{y\in f^{-1}(x)} \chi_k(y) \\
&= \sum_{k,\ell} \frac{1}{|\gamma_k|}\rho_k \sum_{y\in f^{-1}(x)} \chi_k(y)\chi_\ell(x) \qquad \left(\text{since} \quad 1 = \sum_\ell \chi_\ell(x)\right) \\
&= \sum_{k,\ell} \frac{1}{|\gamma_k|}\rho_k A_{k\ell}\chi_\ell(x) \qquad (\text{ since} \quad A_{k\ell} = 1 \Leftrightarrow f(I_k) \supseteq I_\ell) \\
&= \sum_\ell \left(\sum_k \frac{A_{k\ell}}{|\gamma_k|}\rho_k\right) \chi_\ell(x) = \rho(x)
\end{aligned}$$

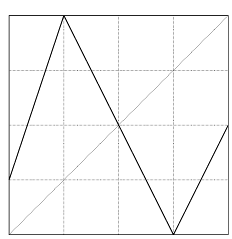
where the last step requires

$$\sum_k \frac{A_{k\ell}}{|\gamma_k|}\rho_k = \sum_k T_{\ell k}\rho_k = \rho_\ell\,.$$

The square matrix $\underline{\underline{T}}$ has an eigenvalue one since the matrix has a left-eigenvector with eigenvalue one (mainly a reformulation of Perron's theorem)

$$f(I_k) = \bigcup_{\ell, A_{k\ell}=1} I_\ell \quad\Rightarrow\quad \underbrace{|f(I_k)|}_{=|\gamma_k||I_k|} = \sum_\ell A_{k\ell}|I_\ell| \quad\Rightarrow\quad |I_k| = \sum_\ell \frac{A_{k\ell}}{|\gamma_k|}|I_\ell| = \sum_\ell |I_\ell| T_{\ell k}\,.$$

b)



Markov partition:

$$I_0 = [0, 1/4], \quad I_1 = [1/4, 1/2], \quad I_2 = [1/2, 3/4] \quad I_4 = [3/4, 1]$$

Transition matrix:

$$\underline{\underline{A}} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Transfer matrix:

$$\underline{\underline{T}} = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 1/3 & 0 & 1/2 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$$

Invariant density $\rho(x) = \sum_\ell \rho_\ell \chi_\ell(x)$

$$\begin{array}{lll} \rho_0 = \rho_2/2 + \rho_3/2 & & \\ \rho_1 = \rho_0/3 + \rho_2/2 + \rho_3/2 & \Rightarrow & \rho_1 = \rho_0/3 + \rho_0 = 4\rho_0/3 \\ \rho_2 = \rho_0/3 + \rho_1/2 & \Rightarrow & \rho_2 = \rho_0/3 + 2\rho_0/3 = \rho_0 \\ \rho_3 = \rho_0/3 + \rho_1/2 & \Rightarrow & \rho_3 = \rho_0/3 + 2\rho_0/3 = \rho_0 \end{array}$$

Normalisation: $1 = (\rho_0 + \rho_1 + \rho_2 + \rho_3)/4, \Rightarrow \rho_0 = 12/13.$

Lyapunov exponent

$$\Lambda = \int_I \ln|f'(x)|\rho(x)dx = \sum_\ell \ln|\gamma_\ell|\rho_\ell|I_\ell| = \frac{3}{13}\ln 3 + \frac{10}{13}\ln 2$$