Problem 2

a) Let $x = r\cos(\varphi)$, $y = r\sin(\varphi)$. Then

$$\begin{aligned}\dot{x} &= \dot{r}\cos(\varphi) - r\dot{\varphi}\sin(\varphi) = -r\cos(\varphi) - r\sin(\varphi)\\ \dot{y} &= \dot{r}\sin(\varphi) + r\dot{\varphi}\cos(\varphi) = r\cos(\varphi) - r\sin(\varphi)\,.\end{aligned}$$

Thus the equations of motion in polar coordinates read (multiply by $\cos(\varphi)$, $\sin(\varphi)$ and add/subtract equations)

$$\dot{r} = -r, \qquad \dot{\varphi} = 1$$

and the flow results in

$$r(t) = e^{-t}r(0), \qquad \varphi(t) = t + \varphi(0)\,.$$

Similarly for the stable node, $\xi = \rho\cos(\psi)$, $\eta = \rho\sin(\psi)$

$$\begin{aligned}\dot{\xi} &= \dot{\rho}\cos(\psi) - \rho\dot{\psi}\sin(\psi) = -\rho\cos(\psi)\\ \dot{\eta} &= \dot{\rho}\sin(\psi) + \rho\dot{\psi}\cos(\psi) = -\rho\sin(\psi)\end{aligned}$$

i.e.

$$\dot{\rho} = -\rho, \qquad \dot{\psi} = 0$$

so that

$$\rho(t) = e^{-t}\rho(0), \qquad \psi(t) = \psi(0)\,.$$

To determine a transformation $(r,\varphi) \mapsto (\rho,\psi)$ which preserves the flow it seems sensible to chose $\rho = r$. Furthermore $\psi(t)$ is constant and $\varphi(t) - t = \varphi(t) + \ln(r(t)/r(0))$ is constant as well, thus the choice $\psi = \varphi + \ln(r)$ will do the job.

b) The transformation defined by $(r,\varphi) \mapsto (\rho = r, \psi = \varphi + \ln(r))$ (for $r \neq 0$) leaves the flow invariant. The transformation is invertible as well (as equations can be solved for r and φ), and it is continuous (for $r \neq 0$). If we define the transformation such that the origin is mapped to the origin (i.e. the fixed point to the fixed point) we just have to show that the mapping is continuous at the origin (as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$, i.e., that the map is a continuous function in Cartesian coordinates). In fact

$$\begin{aligned}\xi &= \rho\cos(\psi) = r\cos(\varphi + \ln(r))\\ &= r\left(\cos(\varphi)\cos(\ln(r)) - \sin(\varphi)\cos(\ln(r))\right)\\ &= x\cos\left(\frac{1}{2}\ln(x^2+y^2)\right) - y\sin\left(\frac{1}{2}\ln(x^2+y^2)\right)\\ \eta &= \rho\sin(\psi) = r\sin(\varphi + \ln(r))\\ &= r\left(\sin(\varphi)\cos(\ln(r)) + \cos(\varphi)\sin(\ln(r))\right)\\ &= y\cos\left(\frac{1}{2}\ln(x^2+y^2)\right) + x\sin\left(\frac{1}{2}\ln(x^2+y^2)\right)\end{aligned}$$

is continuous at $(0,0)$ as the trigonometric functions are bounded.

The transformation is a differential rotation where the angle of rotation depends on the radius in a logarithmic way. It acts like a mixer where the angle of rotation increases if the origin is approached.