Problem 1

a) If $\mathcal{O}(\varepsilon^n)$ denotes terms of order ε^n then

$$\begin{aligned}\dot{z}(t) &= \dot{x}(t) + \varepsilon\frac{\partial h_1}{\partial t} + \varepsilon\frac{\partial h_1}{\partial x}\dot{x}(t) + \mathcal{O}(\varepsilon^2)\\ &= \varepsilon f(x(t),t) + \varepsilon\frac{\partial h_1}{\partial t} + \mathcal{O}(\varepsilon^2)\end{aligned}$$

Introducing

$$\delta f(x,t) = f(x,t) - \bar{f}(x), \text{ i.e. } \int_0^T \delta f(x,t')dt' = 0$$

we have

$$\dot{z}(t) = \varepsilon\bar{f}(x(t)) + \varepsilon\left(\delta f(x(t),t) + \frac{\partial h_1}{\partial t}\right) + \mathcal{O}(\varepsilon^2)$$

Thus

$$h_1(x,t) = -\int_0^t \delta f(x,t')dt'$$

satisfies the periodicity condition $0 = h_1(x,0) = h_1(x,T)$ and given that $x = z + \mathcal{O}(\varepsilon)$ we end up with

$$\dot{z}(t) = \varepsilon\bar{f}(x(t)) + \mathcal{O}(\varepsilon^2) = \varepsilon\bar{f}(z(t)) + \mathcal{O}(\varepsilon^2)$$

b) Using

$$\begin{aligned}u &= x\cos(\omega t) - \dot{x}\frac{\sin(\omega t)}{\omega}\\ v &= x\sin(\omega t) + \dot{x}\frac{\cos(\omega t)}{\omega}\end{aligned}$$

we obtain

$$\begin{aligned}\dot{u} &= \dot{x}\cos(\omega t) - \omega\sin(\omega t)x - \ddot{x}\frac{\sin(\omega t)}{\omega} - \dot{x}\cos(\omega t)\\ &= -\omega\sin(\omega t)x + \frac{\sin(\omega t)}{\omega}\left(\varepsilon(x^2-1)\dot{x} + \omega_0^2 x - \varepsilon h\cos(\omega t)\right)\\ \dot{v} &= \dot{x}\sin(\omega t) + \omega\cos(\omega t)x + \ddot{x}\frac{\cos(\omega t)}{\omega} - \dot{x}\sin(\omega t)\\ &= \omega\cos(\omega t)x + \frac{\cos(\omega)}{\omega}\left(-\varepsilon(x^2-1)\dot{x} - \omega_0^2 x + \varepsilon h\cos(\omega t)\right)\end{aligned}$$

Using the detuning $\varepsilon\Delta = \omega^2 - \omega_0^2$ the system results in

$$\begin{aligned}\dot{u} &= \varepsilon\left(-\frac{\Delta}{\omega}\sin(\omega t)x + \frac{\sin(\omega t)}{\omega}\left((x^2-1)\dot{x} - h\cos(\omega t)\right)\right)\\ \dot{v} &= \varepsilon\left(\frac{\Delta}{\omega}\cos(\omega t)x + \frac{\cos(\omega t)}{\omega}\left(-(x^2-1)\dot{x} + h\cos(\omega t)\right)\right)\end{aligned}$$

where on the right hand side x and $\dot{x}$ have to be replaced by

$$\begin{aligned}x &= u\cos(\omega t) + v\sin(\omega t)\\ \dot{x} &= -\omega\sin(\omega t)u + \omega\cos(\omega t)v\end{aligned}$$

Thus u and v are slow variables and we map apply averaging. The linear and the constant contribution are easy to deal with. Using

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2(x)dx = \frac{1}{2\pi}\int_0^{2\pi}\sin^2(x)dx = \frac{1}{2}$$

and the orthogonality between the different trigonometric functions we have ($T = 2\pi/\omega$)

$$\begin{aligned}
\frac{1}{T}\int_0^T -\frac{\Delta}{\omega}\sin(\omega t)x dt &= -\frac{\Delta}{2\omega} \\
\frac{1}{T}\int_0^T \frac{\Delta}{\omega}\cos(\omega t)x dt &= \frac{\Delta}{2\omega} \\
\frac{1}{T}\int_0^T \frac{\sin(\omega t)}{\omega}(-h\cos(\omega t))dt &= 0 \\
\frac{1}{T}\int_0^T \frac{\cos(\omega t)}{\omega}h\cos(\omega t)dt &= \frac{h}{2\omega}
\end{aligned}$$

We are thus left with the nonlinear contribution

$$(x^2-1)\dot{x} = (u^2\cos^2(\omega t) + v^2\sin^2(\omega t) + 2uv\sin(\omega t)\cos(\omega t) - 1)(-\omega\sin(\omega t)u + \omega\cos(\omega t)v)$$

If we observe that

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2(x)\sin^2(x)dx = \frac{1}{8}, \qquad \frac{1}{2\pi}\int_0^{2\pi}\cos^4(x)dx = \frac{1}{2\pi}\int_0^{2\pi}\sin^4(x)dx = \frac{3}{8}$$

and that all the other integrals involving odd powers of trigonometric functions will vanish we end up with

$$\begin{aligned}
\frac{1}{T}\int_0^T \frac{\sin(\omega t)}{\omega}(x^2-1)\dot{x}dt &= -u^2\frac{u}{8} - v^2\frac{3u}{8} + 2uv\frac{v}{8} + \frac{u}{2} \\
&= \frac{u}{2}\left(1 - \frac{1}{4}(u^2+v^2)\right) \\
\frac{1}{T}\int_0^T -\frac{\cos(\omega t)}{\omega}(x^2-1)\dot{x}dt &= -u^2\frac{3v}{8} - v^2\frac{v}{8} + 2uv\frac{u}{8} + \frac{v}{2} \\
&= \frac{v}{2}\left(1 - \frac{1}{4}(u^2+v^2)\right)
\end{aligned}$$

If we combine all the results the averaged equation of motion reads

$$\begin{aligned}
\dot{u} &= \frac{\varepsilon}{2}\left(-\frac{\Delta}{\omega}v + u\left(1 - \frac{1}{4}(u^2+v^2)\right)\right) \\
\dot{v} &= \frac{\varepsilon}{2}\left(\frac{\Delta}{\omega}u + v\left(1 - \frac{1}{4}(u^2+v^2)\right) + \frac{h}{\omega}\right)
\end{aligned}$$

A simple linear rescaling yields the expression used in the lecture notes with $\sigma = \Delta/\omega$ and $\gamma = -h/(2\omega)$.