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Dynamical Systems

Supplementary notes

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These notes cover some issues of mathematical rigour in basic dynamical systems theory.

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1 Some local analysis

We review some concepts regarding the behaviour of functions in the vicinity of a point.

A **neighbourhood** of a point $x \in \mathbb{R}$ is any open interval[1] containing x. In the mathematics literature the accepted meaning of this term is considerably more general (an open set containing x), but the given definition will suffice for our purpose.

The neighbourhood concept characterises proximity in a concise manner that does not rely on quantitative information. A property that holds in a neighbourhood of a point is said to be **local**. For instance, the sentence 'The function f is bounded in a neighbourhood of x_0' means that there is an open interval containing x_0 whose image under f is a bounded set. We could express the same idea by saying that the function f is 'locally bounded'.

This concept is generalised naturally to higher dimensions: a neighbourhood of $x \in \mathbb{R}^2$ is an open disc containing x; in three dimensions we have an open sphere, etc.

1.1 The Big-O notation

Let f and g be real functions. We write

$$f(x) = O(g(x)) \qquad \text{as} \quad x \to 0$$

to mean that there is a constant C such that $|f(x)| < C|g(x)|$ holds in a neighbourhood of the origin, that is, for all sufficiently small $|x|$.

For example

$$\sin(x) = x - \frac{x^3}{3!} + O(x^5)$$

means that there is a constant C such that, for small enough $|x|$

$$\left| \sin(x) - x + \frac{x^3}{3!} \right| < C \left| x^5 \right|.$$

In fact we can choose $|x| < 1$ and $C = 1/5!$. (Think about it.)

There is an analogous definition for the case $x \to \infty$, and indeed for x tending to any real or complex value. Note that $O(1)$ just means *bounded* in the relevant domain.

[1] An interval is open if it does not contain its end-points

For example, as $x \to 0$, we have

$$x^2 = O(x) \qquad \cos(x) - 1 = O(x^2) \qquad \cos(x) + 1 = O(1),$$
$$\frac{e^x - 1}{x} = 1 + O(x) \qquad \frac{1}{1-x} = 1 + x + O(x^2).$$

As $x \to \infty$, we have

$$x = O(x^2) \qquad \frac{x^2+1}{x^3} = O\left(\frac{1}{x}\right) \qquad \arctan(x) = O(1),$$
$$x\log(x) = O(x^2) \qquad \log(x)^6 = O(\sqrt{x}).$$

The following properties of the big-O notation are useful in computations (here f and g are arbitrary real functions and c is a constant):

$$\begin{aligned} cf(x) &= O(f(x)) \\ O(f(x))O(g(x))) &= O(f(x)g(x)) \\ O(f(x)) + O(g(x)) &= O(|f(x)| + |g(x)|) \\ O(f(x)g(x)) &= f(x)O(g(x)) \\ O(f(x)^2) &= O(f(x))^2. \end{aligned}$$

Let us apply these identities in some examples. Assume that $f(x) = O(1)$ as $x \to 0$. Then

$$e^{O(f(x))} = 1 + O(f(x)) \qquad x \to 0.$$

As $x \to 0$ and $y \to 0$ we have:

$$\begin{aligned} (1+x)^y &= e^{y\log(1+x)} \\ &= 1 + y\log(1+x) + O(y^2\log(1+x)^2) \\ &= 1 + yx + O(yx^2) + O(y^2x^2) \\ &= 1 + yx + O(yx^2). \end{aligned}$$

Thus

$$(1+x)^y = \begin{cases} 1 + yx + O(x^2) & y \text{ fixed} \\ 1 + O(y) & x \text{ fixed.} \end{cases}$$

The big-O notation represents a rare instance in mathematics in which the symbol '=' does not mean equal. Indeed from the statements

$$\sin(x) = O(x) \qquad x^2 = O(x) \qquad x \to 0$$

we cannot deduce that $\sin(x) = x^2$ in a neighbourhood of 0. Here the equal sign really denotes membership: $\sin(x) \in O(x)$, where $O(x)$ is the set of functions which are defined in a neighbourhood of the origin and have the stated property. Thus

$$O(x) = \{x, x^2, \sin(x), \log(1+x), \ldots\} \qquad x \to 0.$$

Exercise 1.1. Prove that

$$1 + 2x + O(x^2) = (1+2x)(1+O(x^2)) \qquad x \to 0.$$

Exercise 1.2. Prove or disprove

1. $\dfrac{1}{1+x^2} = 1 + O(x) \qquad x \to 0$
2. $\cos(x)\sin(x) = O(x^2) \qquad x \to \infty$
3. $\cos(x)\sin(x) = O(x^2) \qquad x \to 0$
4. $\cos(O(x)) = 1 + O(x^2) \qquad$ all x
5. $O(x+y) = O(x^2) + O(y^2) \qquad x, y \to \infty$
6. $e^{(1+O(1/n))^2} = e + O(1/n) \qquad n \to \infty$
7. $n^{\log(n)} = O(\log(n)^n) \qquad n \to \infty$

[*Answer:* T,T,F,T,T,T,T]

Exercise 1.3. Multiply $(\log(n) + \gamma + O(1/n))$ by $(n + O(\sqrt{n}))$ and express your answer in O-notation.
[*Answer:* $n\log(n) + \gamma n + O(\sqrt{n}\log(n))$]

1.2 Local theorems

Let k be a non-negative integer. A real function f is said to be of **class** C^k (written $f \in C^k$) if f is differentiable k times in its domain of definition, and if the k-th derivative is continuous there. Class C^0 mean continuous, while class C^∞ means that the function has continuous derivatives of all orders. Clearly, if $f \in C^{k+1}$ then $f \in C^k$.

For example the function $f(x) = |x|$ is C^0 (but not C^1), $f(x) = |x|^3$ is C^2 (but not C^3), while $f(x) = \sin(x)^2$ is C^∞.

If the stated smoothness property hold only in a domain $U \subset \mathbb{R}$, we make this explicit by writing $f \in C^k(U)$.

1. TAYLOR'S THEOREM. This is the most important theorem in local analysis.

Theorem 1.1. *Suppose that $f : U \to \mathbb{R}$ is a C^{m+1} function defined over an open interval U, and let a and x be any two points in U. Then there is a point ζ between a and x such that*

$$f(x) = \sum_{k=0}^{m} \frac{1}{k!} f^{(k)}(a)(x-a)^k + \frac{1}{(m+1)!} f^{(m+1)}(\zeta)(x-a)^{m+1}$$

where $f^{(k)}$ denotes the k-th derivative of f.

So we have, for any $m \geqslant 0$

$$f(x) = \sum_{k=0}^{m} \frac{1}{k!} f^{(k)}(a)(x-a)^k + O((x-a)^{m+1}) \qquad x \to a.$$

2. THE IMPLICIT FUNCTION THEOREM. In what follows you may regard an open subset of $\mathbb{R}^2$ as an open disc.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be an open set and let $F : \Omega \to \mathbb{R}$ be a differentiable function with continuous partial derivatives. Moreover let $(x_0, y_0) \in \Omega$ be such that*

$$F(x_0, y_0) = c, \qquad \left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0)} \neq 0.$$

Then there exist open intervals U, V with $(x_0, y_0) \subset U \times V \subset \Omega$ and a differentiable function g such that the two sets

$$\{(x,y) \in \Omega : F(x,y) = c\} \quad \text{and} \quad \{(x, g(x)) : x \in U\}$$

agree on $U \times V$.

The theorem says that, provided that $\partial F/\partial y$ is non-zero at (x_0, y_0), then near this point the level set $F(x,y) = c$ may be expressed as the graph of a smooth function $y = y(x) = g(x)$. The role of x and y may be interchanged to obtain an analogous statement for the implicit function $x = x(y)$.

For example, the level set $F(x,y) = x^2 + y^2 = c^2$ (a circle centred at the origin) is the graph of a function $y = g(x)$ as long as $\partial F/\partial y = 2y$ is non-zero, that is, $y \neq 0$. At the points $(\pm c, 0)$ two distinct implicit functions coalesce.

2 Flows on the line

Let $U \subset \mathbb{R}$ be an open interval (possibly infinite, or semi-infinite), and let $v : U \to \mathbb{R}$ be a function (called a **vector field**). A **solution** of the first order ordinary differential equation

$$\dot{x} = v(x) \qquad x \in U \tag{1}$$

where $\dot{x} = \mathrm{d}x/\mathrm{d}t$, is a differentiable function

$$x : I \to U$$

of an open interval $I \subset \mathbb{R}$ (possibly infinite, or semi-infinite) such that

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=\tau} = v(x(\tau)) \tag{2}$$

The value $x_0 = x(0)$ is called the **initial condition**. The initial time $t = 0$ could be replaced by any $t_0 \in \mathbb{R}$.

Let a be a real number. The solution of the differential equation

$$\dot{x} = ax \qquad x(0) = x_0 \in \mathbb{R} \tag{3}$$

is $x(t, x_0) = x_0 e^{at}$. This function is defined over the entire real line.

2.1 Existence and uniqueness

The following theorem gives conditions under which the solution of equation (1) exists and is unique for any choice of initial conditions.

Theorem 2.1. *Let v and U be as above, with $v \in C^1(U)$. Then for every $x_0 \in U$ the following holds:*

i) Equation (1) has a solution $x = x(t)$ with $x(0) = x_0$, valid for all t sufficiently near 0.

ii) This solution is unique, that is, any two solutions $x^{(1)}$ and $x^{(2)}$ coincide in a neighbourhood of $t = 0$.

iii) If $v(x_0) = 0$, then $x(t) \equiv x_0$; if $v(x_0) \neq 0$, then

$$t = \int_{x_0}^{x(t)} \frac{\mathrm{d}z}{v(z)}. \tag{4}$$

If $v(x_0) \neq 0$, then the function $t = t(x)$ defined by equation (4) may be inverted in some neighbourhood of $t = 0$, giving the solution $x = x(t)$. Invertibility follows from the implicit function theorem (see section 1.2) applied to the function

$$F(t,x) = t - \int_{x_0}^{x(t)} \frac{\mathrm{d}z}{v(z)}.$$

Indeed from the fundamental theorem of Calculus, the derivative of F with respect to x is equal to the integrand, and the latter is non-zero by assumption. Thus the expression (4) may be inverted to give a differentiable function $x = x(t)$. We see that solving (1) involves an integration (by separation of variables) and an inversion.

The restrictions appearing in the formulation of the theorem are necessary. Thus the solution $x(t)$ may not be defined for all real t. For instance, the system

$$\dot{x} = x^2 \qquad x(0) = x_0$$

has solution

$$x(t,x_0) = \frac{x_0}{1 - x_0 t}.$$

For $x_0 \neq 0$, this function is not defined over the whole $\mathbb{R}$. It is defined in the interval $(-\infty, 1/x_0)$ for $x_0 > 0$, and $(1/x_0, \infty)$ for $x_0 < 0$.

The differentiability of the vector field is necessary for uniqueness. For example, the system

$$\dot{x} = \sqrt{x} \qquad x(0) = x_0 \qquad x \geqslant 0$$

has solution

$$x(t,x_0) = (t + \sqrt{x_0})^2/4.$$

However, if $x_0 = 0$, then there is also the solution which is identically zero for all t. So the solution is not unique.

2.2 Stability

A point $x^* \in U$ for which $v(x^*) = 0$ is called a **fixed point** (or **equilibrium point**) of the flow.

An equilibrium point x^* is (**Lyapounov**) **stable** if for every neighbourhood V of x^* there is a neighbourhood W of x^* such that all solutions $x = x(t)$ with $x(0) \in W$ remain in V for all times. This means that all points sufficiently close to x^* remain near x^*, although they do not necessarily approach x^*.

Lyapounov stability is **continuity** in disguise. Indeed, let us consider the one-parameter family of functions

$$\varphi_t : U \to U \qquad \varphi_t(x_0) = x(t, x_0).$$

From theorem 2.1 we know that φ_t is defined in U for all sufficiently small t. If the equilibrium point $x^* \in U$ is stable, then, for these values of t, all functions φ_t are continuous at x^*.

An equilibrium point x^* ($v(x^*) = 0$) is **asymptotically stable** if it is stable[2] and if there is a neighbourhood V of x^* such that, for any $x_0 \in V$ all solutions $x = x(t, x_0)$ converge to x^* as $t \to \infty$. The equation (3) has the fixed point $x^* = 0$. If $a < 0$, then $x(t, x_0) \to 0$ for any choice of the initial condition x_0. In this case asymptotic stability is a global property: $V = U = \mathbb{R}$.

A point may be stable but not asymptotically stable, in which case it may be called called **neutrally** (or **marginally**) **stable**. For instance, if $v(x) = 0$, then all points $x^* \in \mathbb{R}$ are fixed and stable, but not asymptotically stable. A more meaningful example is given in exercise 2.1.

The point x^* is **unstable** if there is a neighbourhood V of x^* such that all solutions with $x(0) \in V \setminus \{x^*\}$ eventually leave V. In the system $\dot{x} = x$, the origin is unstable. Note that stable is not the logical negation of unstable, that is, there are fixed points which are neither stable nor unstable, for instance the origin in the system $v(x) = x^2$.

Consider now an equilibrium point x^* for a field v, and suppose that $\lambda = v'(x*) \neq 0$. Without loss of generality, we may assume that $x^* = 0$. Taylor's theorem gives

$$v(x) = \lambda x + g(x) \qquad \text{where} \qquad g(x) = O(x^2).$$

Since $\lambda \neq 0$, we choose ε such that $0 < \varepsilon < |\lambda|$. Now $g'(x) = O(x)$ tends to 0 as x tends to zero, and so there is a neighbourhood U of 0 such that, for all $x \in U$ we have $|g'(x)| < \varepsilon$, and hence (by integration) $|g(x)| < \varepsilon |x|$. This implies that the sign of $v(x)$ is the same as that of its linear part λx. The argument above, together with exercise 2.2 *ii*) below, gives the following result.

Theorem 2.2. *Suppose that v is a C^1 function and that x^* is an equilibrium point. If $v'(x^*) < 0$, then x^* is asymptotically stable; if $v'(x^*) > 0$, then x^* is unstable.*

Thus the stability of an equilibrium point x^* of a C^1 vector field is completely determined by the derivative of the field at x^*, as long as this derivative is non-

[2] Requiring stability is redundant here —see exercise 2.2 *iv*)— but it'll become essential for flows on the circle, or in higher dimensions.

zero. An equilibrium point such that $v'(x^*) \neq 0$ is said to be **hyperbolic**. The non-hyperbolic points play a key role in the description of the parameter-dependence of a vector field, as we shall see in the next section.

Exercise 2.1. Consider the differential equation

$$\dot{x} = v(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin(\log(|x|)) & \text{if } x \neq 0. \end{cases}$$

Show that in the open interval $(-1,1)$, the function v is of class C^1 but not C^2, and that the origin is neutrally stable (stable but not asymptotically stable).
[*Show that* $v''(x) = \sin(\log(|x|)) + 3\cos(\log(|x|))$.]

Exercise 2.2. Let x^* be an equilibrium point of $\dot{x} = v(x)$, on the line.

i) Prove that x^* is stable if there is a neighbourhood U of x^* such that $(x - x^*)v(x) \leqslant 0$ for all $x \in U$.

ii) Prove that x^* is asymptotically stable if there is a neighbourhood U of x^* such that $(x - x^*)v(x) < 0$ for all $x \in U \setminus \{x^*\}$.

iii) Show that in the statement *i*) above, the 'if' could not be replaced by 'if and only if'.

iv) Prove that if all orbits in a neighbourhood of x^* converge to x^*, then x^* is stable.

3 Bifurcations

We consider a family v of vector fields on the line depending smoothly on a real **parameter** r:

$$\dot{x} = v(x,r) \qquad v : \mathbb{R}^2 \to \mathbb{R}. \tag{5}$$

(More generally, the domain of v will be an open domain in $\mathbb{R}^2$.) In the literature one often finds the notation $v_r(x)$ for $v(x,r)$, which emphasises the subordinate role of the argument r (this is why r is called a *parameter* rather than a *variable*). [Even though the co-domain of v is $\mathbb{R}$, not $\mathbb{R}^2$, expression (5) may still be interpreted as defining a vector field on $\mathbb{R}^2$, which is everywhere parallel to the x-axis.]

We require that the field v be a differentiable function of x for every r, so that the existence and uniqueness theorem of section 2 applies. We also require that v

be a differentiable function of r, that is, as the parameter is varied, the vector field changes smoothly. In symbols: $v \in C^1(\mathbb{R}^2)$.

The solution set $\mathscr{B}$ of the fixed point equation $v(x,r) = 0$, given by

$$\mathscr{B} = \{(r,x) \in \mathbb{R}^2 : v(x,r) = 0\} \tag{6}$$

is called the **bifurcation diagram** of the field v. Take a point $(r_0, x_0) \in \mathscr{B}$, representing a fixed point of the vector field v for a particular parameter value r_0. If the condition

$$\left.\frac{\partial v}{\partial x}\right|_{(x_0,r_0)} \neq 0 \tag{7}$$

is satisfied, then the fixed point is hyperbolic. The implicit function theorem then ensures that as r varies near r_0, the location of the fixed point is given by a differentiable function $x^* = x^*(r)$, with $x_0 = x^*(r_0)$. The local function $x^*(r)$ represents a **branch** of the bifurcation diagram.

From theorem 2.2 we know that the stability properties of the point x^* at $r = r_0$ are determined by the sign of the derivative (7). We now show that the same is true for all r sufficiently close to r_0. Indeed we have

$$\left.\frac{\partial v}{\partial x}\right|_{(x_0,r_0)} = \left.\frac{\partial v}{\partial x}\right|_{(x^*(r_0),r_0)} \neq 0,$$

and furthermore, both x^* and v are continuous functions of r. It follows that there is a neighbourhood V of r_0 such that for all $r \in V$ we have

$$\left.\frac{\partial v}{\partial x}\right|_{(x^*(r),r)} \neq 0$$

and moreover the sign of this derivative agrees with that of (7), as desired.

We have shown that the flow near a hyperbolic equilibrium point is insensitive to small perturbations of the vector field. So to obtain qualitative changes of behaviour when a parameter is varied, the spatial derivative of the vector field must vanish. Thus we require the simultaneous conditions:

$$v(x_0, r_0) = 0 \qquad\qquad \left.\frac{\partial v}{\partial x}\right|_{(x_0,r_0)} = 0. \tag{8}$$

A parameter value for which conditions (8) holds is called a **critical** (or **bifurcation**) parameter $r = r_c$. In this case the fixed point x^* is (at least) a **double zero** of v, that is, as $x \to x^*$ we have

$$v(x, r_c) = a(x - x^*)^n + O\left((x - x^*)^{n+1}\right) \qquad a \neq 0, \quad n \geqslant 2.$$

At a bifurcation point the implicit function theorem does not apply. Geometrically this is a point in the bifurcation diagram where several branches of the fixed point function come together. The order n of the zero of v at x^* is given by the number of fixed points that coalesce at the critical parameter.

3.1 Types of bifurcations

We consider three types of bifurcations of fixed points, which are represented by the following one-parameter families of differential equations:

$$\dot{x} = r - x^2 \qquad \text{saddle-node} \tag{9}$$

$$\dot{x} = rx - x^2 \qquad \text{transcritical} \tag{10}$$

$$\dot{x} = rx \pm x^3 \qquad \text{pitchfork} \tag{11}$$

It is immediate to check that in all cases the bifurcation conditions (8) are satisfied at $(x,r) = (0,0)$, which is therefore a bifurcation point.

SADDLE-NODE. For the general system (5), a saddle-node bifurcation at $(x,r) = (x^*, r_c)$ is defined by the following **transversality conditions**, which are clearly verified by (9):

$$\left.\frac{\partial v}{\partial r}\right|_{(x^*,r_c)} \neq 0 \qquad \left.\frac{\partial^2 v}{\partial x^2}\right|_{(x^*,r_c)} \neq 0. \tag{12}$$

The first condition, together with the implicit function theorem, implies that, as r varies, the fixed point of (5) draws a curve which is tangent to the line $r = r_c$. The second condition ensures that the bifurcation curve has a quadratic tangency with $r = r_c$, and locally lies on one side of this line.

With the normal form (9) branching occurs above the bifurcation value where two equilibria, one stable and one unstable, are present, while there are no fixed points below the bifurcation value. The bifurcation diagram for the alternative normal form $v(x,r) = r + x^2$ is obtained from that of (9) via a reflection with respect to the origin in the $r-x$-plane.

TRANSCRITICAL. We see that the equation (10) has the fixed point $x = 0$ for all parameter values. Any system with such a feature cannot go through a saddle-node bifurcation, since the latter requires that there are no fixed points near the bifurcation point. This prevents the first transversality condition in (12) from being satisfied.

We replace this condition by an alternative requirement, which is satisfied by the normal form (10)

$$\left.\frac{\partial^2 v}{\partial r \partial x}\right|_{(x^*,r_c)} \neq 0 \qquad \left.\frac{\partial^2 v}{\partial x^2}\right|_{(x^*,r_c)} \neq 0. \tag{13}$$

PITCHFORK. A vector field which is an odd function of x, namely

$$v(-x,r) = -v(x,r)$$

is said to be **symmetric** or **equivariant**. A symmetric system must have a fixed point at the origin. However, the second condition in (13) cannot be satisfied by an odd function, and so we replace it by the following transversality conditions

$$\left.\frac{\partial^2 v}{\partial r \partial x}\right|_{(x^*,r_c)} \neq 0 \qquad \left.\frac{\partial^3 v}{\partial x^3}\right|_{(x^*,r_c)} \neq 0 \tag{14}$$

which are satisfied by the normal form (11) for both choices of sign.

The sign of the third derivative at bifurcation determines both the direction of bifurcation and the stability properties of all fixed points. If $\partial^3 v/\partial x^3$ is negative, then the branching occurs above the bifurcation value, where two stable and one unstable fixed points are present, while below the bifurcation point there is a single stable equilibrium. In this case we speak of a **supercritical** pitchfork bifurcation.

If the third derivative is positive, then we have a **subcritical** pitchfork bifurcation. The branching occurs below the bifurcation value (two unstable and one stable equilibria), while above bifurcation there is a single unstable fixed point.

4 Flows on the circle

We denote the unit circle by $\mathbb{S}^1$. To define a vector field on $\mathbb{S}^1$ we consider functions $v : \mathbb{R} \to \mathbb{R}$ with are differentiable and 2π**-periodic**, namely:

$$\begin{array}{ll} i) & v \in C^1(\mathbb{R}) \\ ii) & \forall x \in \mathbb{R},\ v(x+2\pi) = v(x). \end{array}$$

Then for any $x \in \mathbb{R}$ and $k \in \mathbb{Z}$ we have $v(x) = v(x+2k\pi)$ and hence $v'(x) = v'(x+2k\pi)$. Such a function defines a vector field on $\mathbb{S}^1$.

On the circle we must refine the notion of stability. In section 2.2 we introduced two definitions of stability: Lyapounov stability (all orbits near a fixed point x^* remain near it for all times) and the stronger asymptotic stability (all orbits near a fixed point converge to it).

On the circle a new phenomenon emerges. Consider the system $\dot{x} = 1 - \cos(x)$ on $\mathbb{S}^1$. The fixed point $x^* = 0$ is not stable (hence not asymptotically stable), yet all orbits converge to it. We call such a point **attracting** —to differentiate it from asymptotically stable. More precisely, a fixed point x^* is attracting if there is a neighbourhood U of x^* such that, for any $x_0 \in U$ all solutions $x = x(t, x_0)$ converge to x^* as $t \to \infty$. We see that an asymptotically stable point is also attracting, but not vice-versa.

4.1 Periodic functions

We review some methods for constructing periodic functions.

The quintessential 2π-periodic functions are the sine and the cosine, which are C^∞, and for any $k \in \mathbb{N}$, the functions

$$c_k(x) = \cos(kx) \qquad s_k(x) = \sin(kx)$$

have the same properties. Therefore any finite linear combination of these functions

$$v(x) = a_0 + \sum_{k=1}^{n} a_k c_k(x) + b_k s_k(x) \tag{15}$$

where a_k, b_k are arbitrary real numbers, will also have the same properties.

Powers of sines and cosines are also C^∞ and 2π-periodic. Moreover, these functions are rational linear combinations of c_k and s_k. For example:

$$\begin{aligned} \sin^2(x) &= \frac{1}{2} - \frac{1}{2}c_2(x) \\ \sin^3(x) &= \frac{3}{4}s_1(x) - \frac{3}{4}s_3(x) \\ \sin^4(x) &= \frac{3}{8} - \frac{1}{2}c_2(x) + \frac{1}{8}c_4(x). \end{aligned}$$

It follows that any polynomial in c_k and s_k will be of the form (15).

A useful device for constructing periodic functions with a prescribed smoothness is to consider the interval $I = (-\pi, \pi]$ (or any half-open interval of length 2π) and any function $f : I \to \mathbb{R}$ with the property that $f(-\pi) = f(\pi)$ and $f'(-\pi) =$

$f'(\pi)$. Then we **extend** f **periodically** to a function v defined over the entire real line, by letting $v(x) = f(\bar{x})$, where $\bar{x}$ is the unique element of the set $I \cap x + 2\pi\mathbb{Z}$. (Think about it.) The smoothness of v doesn't depends only on the smoothness of f, but also —crucially— on the behaviour of f at the end-points of I.

For example, let us consider the C^∞ function

$$f : (-\pi, \pi] \to \mathbb{R} \qquad f(x) = x(\pi^2 - x^2). \tag{16}$$

We verify that $f(\pi) = f(-\pi) = 0$, and $f'(\pi) = f'(-\pi) = -2\pi^2$. However, $f''(\pi) = -6\pi$ but $f''(-\pi) = 6\pi$. Thus the periodic extension v of f is C^1 but not C^2.

The theory of periodic functions requires considering the limit $n \to \infty$ in (15). The resulting series is called a **Fourier series**, and any non-pathological periodic function will admit such a representation. The questions of convergence and properties of the sum of a Fourier series lie beyond the scope of this course. Here we merely note that any function which is not of class C^∞, such as the piecewise cubic function given above, will necessarily be represented by an infinite Fourier series.

Exercise 4.1. Define a vector field on the circle which is of class C^2 but not C^3. (Choose an even quartic polynomial.)

Exercise 4.2. Show that the function v in (15) is even (odd, respectively) precisely if all coefficients b_k (a_k, respectively) are zero.

Exercise 4.3. Show that nothing is gained by extending the sum (15) to negative indices.

Exercise 4.4. Compute the coefficients of the Fourier series of the periodic extension of the function (16).

5 Some linear algebra

The study of first-order linear ordinary differential equations on the plane is an exercise in linear algebra. In preparation for section 6 we review some necessary material.

Two vectors v_1 and v_2 in $\mathbb{R}^2$ are said to be **linearly independent** if the equation

$$xv_1 + yv_2 = \mathbf{0}$$

admits only the trivial solution $x = y = 0$.

Let $\mathbf{A}$ be a 2×2 real matrix:

$$\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a,b,c,d\in\mathbb{R}. \tag{17}$$

The quantities

$$\operatorname{Tr}(\mathbf{A})=a+d \qquad \operatorname{Det}(\mathbf{A})=ad-bc$$

are called the **trace** and the **determinant** of $\mathbf{A}$, respectively.

The matrix $\mathbf{A}$ is **invertible** if there is a 2×2 real matrix A^{-1} such that $\mathbf{A}\mathbf{A}^{-1}=\mathbf{A}^{-1}\mathbf{A}=\mathbb{1}$, where $\mathbb{1}$ is the 2×2 identity matrix. A matrix is invertible if and only if its determinant is non-zero.

A (real or complex) number λ is called an **eigenvalue** of a matrix $\mathbf{A}$ if there is a (real or complex) non-zero vector v such that $\mathbf{A}v=\lambda v$, that is, $(\mathbf{A}-\lambda\mathbb{1})v=\mathbf{0}$. The vector v is called an **eigenvector** of $\mathbf{A}$ corresponding to the eigenvalue λ. The eigenvectors corresponding to a given eigenvalue form a linear space (the kernel of the matrix $\mathbf{A}-\lambda\mathbb{1}$), and two eigenvectors corresponding to distinct eigenvalues are linearly independent. (The converse, however, is not true.)

The eigenvalues of a matrix $\mathbf{A}$ are the roots of the quadratic polynomial

$$\operatorname{Det}(\mathbf{A}-\lambda\mathbb{1})=\lambda^2-\operatorname{Tr}(\mathbf{A})\lambda+\operatorname{Det}(\mathbf{A})=0, \tag{18}$$

called the **characteristic polynomial** of $\mathbf{A}$.

For the purpose of classifying matrices, we introduce three special parametrised families of matrices, called **Jordan canonical matrices**:

$$\mathbf{J}^{(1)}=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \mathbf{J}^{(2)}=\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \mathbf{J}^{(3)}=\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \tag{19}$$

Where $\lambda_1,\lambda_2,\lambda,\alpha,\beta\in\mathbb{R}$ and $\beta\neq 0$. The following theorem explains why Jordan canonical matrices are important.

Theorem 5.1. *Let $\mathbf{A}$ be a 2×2 real matrix. Then there is an invertible 2×2 real matrix $\mathbf{P}$ and a Jordan canonical matrix $\mathbf{J}$ such that*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P}=\mathbf{J}. \tag{20}$$

Moreover, the matrices $\mathbf{A}$ and $\mathbf{J}$ have the same eigenvalues.

The matrix $\mathbf{J}$ appearing in (20) is called the **Jordan canonical form** of the matrix $\mathbf{A}$. This theorem says that if we change co-ordinates in an appropriate way, then any matrix will be transformed into precisely one of the three Jordan matrices (19).

The following statements are readily verified:

1. The matrix $\mathbf{J}^{(1)}$ has two real eigenvalues λ_1 and λ_2, with corresponding eigenvectors $v_1 = \binom{1}{0}$ and $v_2 = \binom{0}{1}$.
2. The matrix $\mathbf{J}^{(2)}$ has a single real eigenvalue λ with eigenvector v_1.
3. The matrix $\mathbf{J}^{(3)}$ has two complex conjugate eigenvalues $\lambda_\pm = \alpha \pm \mathrm{i}\beta$ with eigenvectors $v_\pm = \binom{1}{\pm\mathrm{i}}$.

To determine the Jordan form of $\mathbf{A}$, we compute the eigenvalues of $\mathbf{A}$. If they are real and distinct then the Jordan form is $\mathbf{J}^{(1)}$; if they are complex then the Jordan form is $\mathbf{J}^{(3)}$. If the eigenvalues are equal, then the Jordan form is $\mathbf{J}^{(1)}$ if there are two linearly independent eigenvectors (hence any vector is an eigenvector and $\mathbf{A}$ is already in Jordan form) and $\mathbf{J}^{(2)}$ otherwise.

Once we have identified the Jordan form, the following result allows us to compute the matrix $\mathbf{P}$ of equation (20). In what follows we denote by $(v_1|v_2)$ the matrix whose columns are the column vectors v_1 and v_2.

Theorem 5.2. *Let $\mathbf{A}$, $\mathbf{J}$, and $\mathbf{P} = (v_1|v_2)$ be as in theorem 5.1. Then the vectors v_1 and v_2 are determined as follows:*

1. *If $\mathbf{J} = \mathbf{J}^{(1)}$ then v_1 and v_2 are any two linearly independent eigenvectors of $\mathbf{A}$.*
2. *If $\mathbf{J} = \mathbf{J}^{(2)}$ then v_1 is an eigenvector of $\mathbf{A}$ and v_2 satisfies the equation*

$$(\mathbf{A} - \lambda \mathbb{1})v_2 = v_1.$$

3. *If $\mathbf{J} = \mathbf{J}^{(3)}$ then v_1 and v_2 are the real and imaginary part of a complex eigenvector of $\mathbf{A}$.*

To see why this theorem works, first note that in all cases the columns of $\mathbf{P} = (v_1|v_2)$ are linearly independent (see exercise 5.3), so that $\mathbf{P}$ is invertible. We compute:

$$\begin{aligned}
\mathbf{J} = \mathbf{J}^{(1)}: &\quad \mathbf{AP} = (\mathbf{A}v_1|\mathbf{A}v_2) = (\lambda_1 v_1|\lambda_2 v_2) = \mathbf{PJ}^{(1)} \\
\mathbf{J} = \mathbf{J}^{(2)}: &\quad \mathbf{AP} = (\mathbf{A}v_1|\mathbf{A}v_2) = (\lambda v_1|v_1 + \lambda v_2) = \mathbf{PJ}^{(2)} \\
\mathbf{J} = \mathbf{J}^{(3)}: &\quad \mathbf{AP} = (\mathbf{A}v_1|\mathbf{A}v_2) = (\alpha v_1 - \beta v_2|\beta v_1 + \alpha v_2) = \mathbf{PJ}^{(3)}.
\end{aligned}$$

[In the last expression, we have used $\mathbf{A}v = \mathbf{A}(v_1 + \mathrm{i}v_2) = (\alpha + \mathrm{i}\beta)(v_1 + \mathrm{i}v_2) = (\alpha v_1 - \beta v_2) + \mathrm{i}(\beta v_1 + \alpha v_2)$.] Thus, in each case $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{J}$.

Exercise 5.1. Using the notation of theorems 5.1 and 5.2, compute **J** and **P** for the following matrices **A**:

$$\begin{pmatrix} 5 & -4 \\ 4 & -5 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -\omega^2 & 2\alpha \end{pmatrix}$$

where α and ω are real constants (you'll have to consider their relative magnitude.) In each case verify that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{J}$.

Exercise 5.2. Prove that any two conjugate matrices, that is, two matrices **A** and **J** related as in equation (20), have the same characteristic polynomial.

Exercise 5.3. Prove that the vectors v_1 and v_2 of theorem 5.2 are linearly independent in each case.
[*In case 2, show that* $(\mathbf{A} - \lambda \mathbb{1})^2 v_2 = \mathbf{0}$, *that is,* v_2 *is a* ***generalised eigenvector***.]

6 Linear planar systems

We begin out study of first-order ordinary differential equations on the plane with the linear systems:

$$\dot{z} = \mathbf{A}z \qquad z = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{21}$$

where $a, b, c, d \in \mathbb{R}$. (In this section, the points $z \in \mathbb{R}^2$ are represented as column vectors.) The vector field $\dot{z}$ is a linear function of the co-ordinates. Since $A\mathbf{0} = \mathbf{0}$, all linear systems have a fixed point at the origin.

The linearity of $\dot{z} = \mathbf{A}z$ implies that if $z_1(t)$ and $z_2(t)$ are solutions of (21) then so is any linear combination of z_1 and z_2:

$$z(t) = c_1 z_1(t) + c_2 z_2(t)$$

where c_1 and c_2 are arbitrary real numbers. This is called the **superposition principle**, which says that the solution set of the system (21) is a **real vector space**. To determine its dimension, we must generalise the notion of linear independence.

Two solutions $z_1(t)$ and $z_2(t)$ of (21) are said to be **linearly independent** if, for all $t \in \mathbb{R}$ the relation $c_1 z_1(t) + c_2 z_2(t)$ implies that $c_1 = 0$ and $c_2 = 0$. (We have implicitly assumed that the solutions of a linear system are defined for all real t; this is indeed the case —see exercise 6.3.)

If $z_1(t)$ and $z_2(t)$ are two linearly independent solutions, then the matrix $\mathbf{Z}(t)$ having these vectors as columns, namely

$$\mathbf{Z}(t) = (z_1(t)\,|\,z_2(t))$$

is called a **fundamental matrix solution** of (21). By definition, $\mathbf{Z}(t)$ is invertible for all $t \in \mathbb{R}$, so we let

$$z(t) = \mathbf{Z}(t)\mathbf{Z}(0)^{-1}z_0. \tag{22}$$

We claim that $z(t)$ is the solution of (21) for the initial condition $z(0) = z_0$. Indeed the right hand side of (22) is a solution, being a linear combination of two solutions. Furthermore $z(0) = \mathbf{Z}(0)\mathbf{Z}(0)^{-1}z_0 = z_0$. The result follows from the uniqueness of the solution of the system (21).

Thus the solution space of (21) is two-dimensional, and our task is to find, for given $\mathbf{A}$, a fundamental matrix solution. The simplest cases are the Jordan canonical systems. We have three cases:

$$\begin{aligned}
\mathbf{J} &= \mathbf{J}^{(1)} \qquad \mathbf{Z}(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \\
\mathbf{J} &= \mathbf{J}^{(2)} \qquad \mathbf{Z}(t) = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \\
\mathbf{J} &= \mathbf{J}^{(3)} \qquad \mathbf{Z}(t) = e^{\alpha t}\begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}.
\end{aligned} \tag{23}$$

The validity of these formulae can be verified by a direct calculation (exercise 6.1). Note that $\mathbf{Z}(0) = \mathbb{1}$ in each case.

To solve the linear system (21) with initial conditions $z(0) = z_0$, we proceed as follows. First we compute the eigenvalues and eigenvectors of $\mathbf{A}$, and identify the Jordan canonical form $\mathbf{J}$ of $\mathbf{A}$ as described in section 5. Then we compute the matrix $\mathbf{P}$ using theorem 5.2.

From (20), the co-ordinate change $z = \mathbf{P}w$ transforms the differential equation (21) into

$$\dot{w} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}w = \mathbf{J}w \tag{24}$$

while the initial condition z_0 is changed to w_0, where

$$w_0 = \begin{pmatrix} w_{0,1} \\ w_{0,2} \end{pmatrix} = \mathbf{P}^{-1}z_0.$$

Note that $w_{0,1}$ and $w_{0,2}$ are the components of z_0 with respect to the basis v_1, v_2:

$$z_0 = \mathbf{P}w_0 = (v_1|v_2)w_0 = w_{0,1}v_1 + w_{0,2}v_2.$$

Next we solve (24) using formulae (22) and (23), to obtain $w = w(t)$. Finally, we revert back to the original co-ordinates: $z(t) = \mathbf{P}w(t)$.

Putting everything together, we obtain

$$z(t) = \mathbf{P}w(t) = \mathbf{P}\mathbf{Z}(t)w_0 = \mathbf{P}\mathbf{Z}(t)\mathbf{P}^{-1}z_0.$$

For instance, for $\mathbf{J} = \mathbf{J}^{(1)}$ we have, explicitly:

$$z(t) = w_{0,1}e^{\lambda_1 t}v_1 + w_{0,2}e^{\lambda_2 t}v_2$$

where v_1 and v_2 are two eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues λ_1 and λ_2, respectively.

Exercise 6.1. Verify the validity of formulae (23).

Exercise 6.2. (*The matrix exponential.*) For any square matrix $\mathbf{A}$ and $t \in \mathbb{R}$, define

$$e^{\mathbf{A}t} = \sum_{k \geqslant 0} \frac{1}{k!}\mathbf{A}^k = \mathbb{1} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \cdots .$$

For each of the matrices $\mathbf{J}$ in (23), verify that[3]

$$e^{\mathbf{J}t} = \mathbf{Z}(t)\mathbf{Z}(0)^{-1}.$$

(Comparison with (22) shows that the equation $\dot{z} = \mathbf{J}z$ has solution $z(t) = e^{\mathbf{J}t}z_0$, just like in the one-dimensional case!)

Exercise 6.3. Prove that the solution of (21) is defined for all $t \in \mathbb{R}$.

Exercise 6.4. Let τ and δ be, respectively, the trace and the determinant of a matrix A. Explain what happens to the eigenvalues of A if τ is kept fixed while δ varies in $\mathbb{R}$. Define a one-parameter family A_δ of matrices with fixed trace and varying determinant, and describe the bifurcations of the corresponding system (21). Do the same fixing the determinant and varying the trace.

7 Phase plane

On the Cartesian plane $\mathbb{R}^2$ we measure distances using the Euclidean norm

$$\|z\| = \sqrt{x^2 + y^2}$$

[3] Use the following result: if two matrices $\mathbf{A}$ and $\mathbf{B}$ commute, then $e^{(\mathbf{A}+\mathbf{B})t} = e^{\mathbf{A}t}e^{\mathbf{B}t}$

which satisfies the following properties, for any $z, w \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$:

$$\begin{aligned} &\|z\| \geqslant 0 \qquad \text{and} \qquad \|z\| = 0 \quad \text{iff} \quad z = (0,0) \\ &\|z+w\| \leqslant \|z\| + \|w\| \\ &\|\alpha z\| = |\alpha| \|z\|. \end{aligned}$$

The distance between $z, w \in \mathbb{R}^2$ is defined to be $\|z - w\|$.

We consider planar differential equations of the form

$$\dot{z} = v(z) \qquad z(0) = z_0 \tag{25}$$

where

$$z = (x, y), \qquad v(z) = (v_1(x,y), v_2(x,y)),$$

and v_1 and v_2 are real-valued functions of class C^1. Under this assumption, then, for any $z_0 \in \mathbb{R}^2$ the solution $z = z(t, z_0)$ of the initial value problem (25) exists and is unique in a suitable time-interval $I = I(z_0) = (\alpha(z_0), \beta(z_0))$ containing the origin.

A solution $z(t)$ is **periodic** if there is a positive real number T such that, for all $t \in \mathbb{R}$ we have $z(t) = z(t+T)$. The smallest such a T is called the **period** of the solution. The orbit $\{z(t) : t \in \mathbb{R}\}$ is called a periodic (or closed) orbit.

A point $z^* \in \mathbb{R}^2$ is an **equilibrium point** (or a **fixed point**), if $v(z^*) = 0$, that is, if $z^* = (x^*, y^*)$. This means

$$x^* = v_1(x^*, y^*) \qquad y^* = v_2(x^*, y^*).$$

Let z^* and w^* be distinct equilibria. An orbit is **heteroclinic** if

$$\lim_{t \to -\infty} z(t) = z^* \qquad \lim_{t \to \infty} z(t) = w^*.$$

If $z^* = w^*$, then the orbit is said to be **homoclinic**.

The matrix

$$Dv(z) = \begin{pmatrix} \dfrac{\partial v_1}{\partial x}(z) & \dfrac{\partial v_1}{\partial y}(z) \\ \dfrac{\partial v_2}{\partial x}(z) & \dfrac{\partial v_2}{\partial y}(z) \end{pmatrix} \tag{26}$$

is called the **Jacobian matrix** of v at z. If z^* is an equilibrium point of (25), then the ODE

$$\dot{z} = Dv(z^*) z$$

is called the **linearisation** of the system (25). The error term in approximating (25) by its linearisation is $O(z^2) = (O(x^2, y^2, xy), O(x^2, y^2, xy))$.

7.1 Conservative and reversible systems

A real-valued function $H:\mathbb{R}^2 \to R$ is a **constant of the motion** (or **first integral**) of the system (25) if it is constant along any solution, namely if $H(z(t)) = H(z(0))$ for all t for which the solution is defined. We also require H to be non-trivial, that is, not constant on any open subset of $\mathbb{R}^2$.

For H to be a constant of the motion, the time-derivative of H along the orbits must be equal to zero. We can check this condition without knowledge of the solutions. Indeed let ∇H be the gradient of H. We find:

$$\dot{H}(z) = \frac{\partial H}{\partial x}(z)\dot{x} + \frac{\partial H}{\partial y}(z)\dot{y} = \nabla H(z) \cdot v(z).$$

So the time-invariance of H is equivalent to the condition

$$\nabla H(z) \cdot v(z) = 0. \tag{27}$$

If H is a constant of the motion, then the plane is partitioned into level set of the function H,

$$\{(x,y) \in \mathbb{R}^2 : H(x,y) = const.\},$$

each level set consisting of one or more orbits.

The **conservative systems** are a prominent example of ODEs with a constant of the motion. We start with Newton's law: $F = ma = m\ddot{x}$. Letting $y = \dot{x}$, we transform this second-order ODE into a pair of first-order ODEs:

$$\dot{x} = y \qquad \dot{y} = \frac{1}{m}F(x).$$

Let $V(x)$ be defined —up to an additive constant— by the equation $F(x) = -\mathrm{d}V/\mathrm{d}x$. Such a function is called the **potential**. Then the quantity

$$E(x,y) = \frac{m}{2}y^2 + V(x)$$

called the **energy**, is a constant of the motion. Indeed, from (27) we have

$$\dot{E}(x,y) = (-F(x), my) \cdot (y, F(x)/m) = 0.$$

If $V(x) = x^2$, then the level sets of the energy are ellipses.

A map $R:\mathbb{R}^2 \to \mathbb{R}^2$ is an **involution** if $R^2 = \mathbb{1}$. A system of ODEs is **reversible** if it is invariant under the transformations

$$t \mapsto -t \qquad z \mapsto R(z)$$

for some involution R.

The map $(x,y) \to (x,-y)$ is an involution. Thus the system of equations

$$\dot{x} = v_1(x,y) \qquad \dot{y} = v_2(x,y)$$

is reversible whenever v_1 is an odd function of y and v_2 is an even function of x. All conservative systems are of this form, and hence are reversible.

Exercise 7.1. Consider the potential

$$V_\lambda(x) = \lambda x + x^3.$$

sketch some level sets of E, for different values of λ. Find an equation for the level set that separates closed and non-closed orbits.

Exercise 7.2. Let z^* be an isolated fixed point of a conservative system. Show that if the energy E has a local minimum at z^*, then the level sets of E near z^* are closed curves.

Exercise 7.3. Show that the Jacobian matrix of an involution has determinant ± 1.

8 Limit cycles

Let $z(t,z_0)$ be the solution of (25) with initial condition $z(0) = z_0$, and assume that this solution is defined in a time-interval $I = I(z_0) = (t_-(z_0), t_+(z_0))$ containing the origin. The **positive orbit** $\gamma^+(z_0)$, **negative orbit** $\gamma^-(z_0)$ and the **orbit** $\gamma(z_0)$ of z_0 are defined as

$$\begin{aligned} \gamma^+(z_0) &= \bigcup_{0 \leqslant t < t_+} z(t,z_0) \\ \gamma^-(z_0) &= \bigcup_{t_- < t \leqslant 0} z(t,z_0) \qquad (28) \\ \gamma(z_0) &= \bigcup_{t_- < t < t_+} z(t,z_0) \end{aligned}$$

(29)

If $\gamma(z_0)$ is periodic, then $I = (-\infty, \infty)$ and $\gamma^+ = \gamma^- = \gamma$ is a closed curve (which reduces to a point if z_0 is a fixed point). A non-periodic orbit may approach a fixed point z^*:

$$\lim_{t \to t^+} z(t,z_0) = z^*. \qquad (30)$$

However, a non-periodic orbit may also approach a **limit cycle**, which is an isolated periodic orbit, or a set consisting of several orbits. For example, consider the following systems, expressed in polar co-ordinates (r, θ):

$$i) \quad \begin{array}{rcl} \dot{r} & = & r(1-r) \\ \dot{\theta} & = & 1 \end{array} \qquad\qquad ii) \quad \begin{array}{rcl} \dot{r} & = & r(1-r) \\ \dot{\theta} & = & 1-\cos(\theta)+(r-1)^2. \end{array} \tag{31}$$

All orbits of each system approach the unit circle (apart from the origin). In $i)$ the unit circle is a periodic orbit; in $ii)$ the unit circle is the union of a fixed point and a homoclinic orbit.

We now introduce a machinery to characterise this form of convergence. The main problem to be dealt with is the absence of the limit (30).

A point w is an **ω-limit point** of the orbit $\gamma(z_0)$ if there is a sequence (t_j) of times such that

$$\lim_{j\to\infty} t_j = t_+ \qquad \text{and} \qquad \lim_{j\to\infty} z(t_j, z_0) = w. \tag{32}$$

In other words, w is an ω-limit point if for any neighbourhood U of w there is a time $t^* = t^*(U)$ such that $z(t^*, z_0) \in U$.

The set of all ω-limit points of $\gamma(z_0)$ is called the **ω-limit set** of $\gamma(z_0)$, denoted by $\omega(z_0)$. By reversing the direction of time, and replacing t_+ by t_-, we obtain the analogous concept of **α-limit set**.

The simplest situation occurs when the limit (30) exists. Then, by letting $t_j = j$ and $w = z^*$ in (32), we find that $z^* \in \omega(z_0)$. Since there cannot be any other ω-limit point (see exercises), we have $\omega(z_0) = \{z^*\}$. Thus if z^* is an attractor, all points z_0 in the basin of attraction will have z^* as their common ω-limit set.

The same holds for all points of a homoclinic orbit, whereas if z_0 is heteroclinic we have $\omega(z_0) = \{z^*\}$ and $\alpha(z_0) = \{w^*\}$, where z^* and w^* are two distinct equilibria.

Let Γ be a periodic orbit. If $z_0 \in \Gamma$, then $\alpha(z_0) = \omega(z_0) = \Gamma$. However, if $z_0 \notin \Gamma$, then it is possible that one of the sets $\alpha(z_0)$ and $\omega(z_0)$ is equal to Γ and the other isn't, as in example (31) $i)$. Finally, the possibility exists that $\omega(z_0) = \Gamma$ for $z_0 \notin \Gamma$ but $\omega(z_0) \neq \Gamma$ for $z_0 \in \Gamma$. We saw this in example (31) $ii)$.

Exercise 8.1. Show that if (30) holds, then z^* is the only element of $\omega(z_0)$.

8.1 Lyapounov functions

Let U be an open subset of $\mathbb{R}^2$ containing the origin. A real-valued C^1 function $V : U \to \mathbb{R}^2$ is said to be **positive definite** on U if

$$\begin{array}{lll} i) & V(0,0) = 0 & \\ ii) & V(x,y) > 0 & \forall (x,y) \in U \setminus \{(0,0)\}. \end{array}$$

A homogeneous quadratic form $V(x,y) = ax^2 + bxy + cy^2$ is positive definite on $\mathbb{R}^2$ if and only if $a > 0$ and $b^2 - 4ac < 0$. Indeed, suppose V is positive definite. Since $V(x,0) > 0$ for $x \neq 0$, we must have $a > 0$. If $y = y_0 \neq 0$ is fixed, then there can be no real zero x of $V(x,y_0) = ax^2 + bxy_0 + cy_0^2$. So the discriminant of this polynomial must be negative: $y_0^2(b^2 - 4ac) < 0$, that is $b^2 - 4ac < 0$. Sufficiency follows from similar reasoning —see exercises.

Consider now a systems $\dot{z} = v(z)$, where the vector field v has a stationary point at the origin. A positive-definite function $\mathscr{L}$ of an open neighbourhood U of the origin is said to be a **Lyapounov function** for the field v if

$$\dot{\mathscr{L}}(z) \leqslant 0 \qquad \forall z \in U \setminus \{\mathbf{0}\}. \tag{33}$$

If in (33) we have strict inequality, then we speak of a **strict Lyapounov function**.

The following theorem is due to Lyapounov.

Theorem 8.1. *If $\mathscr{L}$ is a Lyapounov function for a vector field with a fixed point at the origin, then the origin is Lyapounov stable. If $\mathscr{L}$ is a strict Lyapounov function, then the origin is asymptotically stable.*

PROOF. Let U_ε be the open disc of radius ε centred at the origin. Choose ε so that $U_\varepsilon \subset U$. Since $\mathscr{L}$ is continuous, and the boundary of U_ε (a circle) is closed and bounded, then $\mathscr{L}$ assume a minimum value m on such a boundary. Furthermore $m > 0$, because $\varepsilon > 0$ and $\mathscr{L}$ is positive definite.

Now choose δ with $0 < \delta \leqslant \varepsilon$ so that $\mathscr{L}(x) < m$ for $z \in U_\delta$. Such a δ exists because $\mathscr{L}$ is continuous and $\mathscr{L}(\mathbf{0}) = 0$. If $z_0 \in U_\delta$, then for all $t > 0$ we have $z(t,z_0) \in U_\varepsilon$, because $\dot{\mathscr{L}}(z(t)) \leqslant 0$ implies $\mathscr{L}(z(t)) \leqslant \mathscr{L}(z(0))$. This proves that the origin is stable. The proof of asymptotic stability in the strict case is left as an exercise. □

The existence of a strict Lyapounov function implies that there cannot be periodic orbits in U. This is because a strict Lyapounov function is strictly decreasing along orbits, while along a periodic orbit the value of such a function would have to be periodic.

Exercise 8.2. Show that if $a > 0$ and $b^2 - 4ac < 0$, then the quadratic form $ax^2 + bxy + cy^2$ is positive definite.

Exercise 8.3. Complete the proof of theorem 8.1.

8.2 The Poincarè-Bendixon theorem

The following theorem describes the possible limit sets of a planar system.

Theorem 8.2. *Let $\dot{z} = v(z)$ be a planar system with a finite number of equilibrium points. If the orbit $\gamma^+(z_0)$ of z_0 is bounded, then one of the following is true:*

i) The ω-limit set $\omega(z_0)$ is a single equilibrium point z^* and $z(t, z_0) \to z^*$ as $t \to \infty$.

ii) $\omega(z_0)$ is a periodic orbit Γ and the orbit $\gamma^+(z_0)$ is either equal to Γ, or it spirals towards Γ on one side of it.

iii) $\omega(z_0)$ consists of equilibrium points and orbits whose α- and ω-limit sets are equilibrium points.

The following theorem gives sufficient conditions for case *ii*).

Theorem 8.3. (Poincarè-Bendixon) *Any bounded ω-limit set which contains no equilibrium points is a periodic orbit.*

How can we use this theorem to establish the existence of a limit cycle? We begin by defining a **positively invariant set** $\mathscr{D}$ as a set which contains the forward orbit of all its points:

$$\forall z_0 \in \mathscr{D}, \quad \gamma^+(z_0) \subset \mathscr{D}.$$

(Such a set is sometimes called a **trapping region**.) We now attempt to construct a bounded positively invariant set $\mathscr{D}$ which contains no equilibrium point, in the interior or on the boundary. Then the Poincarè-Bendixon theorem ensures that there is at least one cycle in $\mathscr{D}$ (which could be on the boundary).

We illustrate the above approach with an example:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + y(1 - x^2 - 2y^2). \end{aligned} \tag{34}$$

We note that the origin is the only equilibrium point. To determine a positively invariant bounded region $\mathscr{D}$ we consider the function $V(x,y) = x^2 + y^2$, and compute its derivative along the orbits of our system:

$$\dot{V}(x,y) = \nabla V \cdot \dot{z} = 2y^2(1 - x^2 - 2y^2).$$

We verify that if $V(x,y) < 1/2$ then $\dot{V}(x,y) > 0$ and if $V(x,y) > 1$ then $\dot{V}(x,y) < 0$. Therefore for any $1/2 < \varepsilon < 1$, the open annulus

$$\mathscr{D} = \{(x,y) \in \mathbb{R}^2 : \frac{1}{2} - \varepsilon < V(x,y) < 1 + \varepsilon\}$$

has the desired properties; furthermore, the origin is not on its boundary. The Poincarè-Bendixon theorem now implies that there is at least one periodic orbit in $\mathscr{D}$.