MAS/103 Computational Mathematics I: Coursework 8

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This coursework will be assessed and count towards your final mark for the course

DEADLINE: Wednesday of week 12, at 1:00 pm.

CONTENT: Vector spaces.

PREREQUISITES: Chapter 9 of lecture notes.

MicroESSAY : Explain what is Maple. Use approximately 50 words, and no mathematical or Maple symbol.

Problem 1. Load the `linalg` package.

(a) Consider the following vectors in $\mathbf{Z}^3$

$$u = (88, -2, 33), \qquad v = (-7, 2^7, 1).$$

Compute their sum and their scalar product. Determine which of u or v is the longest (using the 2-norm — see lecture notes).

(b) Construct the following matrices

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 0 \\ 3 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 6 & 2 \end{pmatrix}.$$

Then modify one entry of A as follows (without redefining the entire matrix!)

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 0 \\ -5 & 1 \end{pmatrix} \tag{1}$$

Then, using (1), compute BA, $A - B^T$ and $A^T + B$ (the superscript T denotes the transpose of a matrix — see `?transpose`).

Problem 2. Consider the recursive sequence of matrices

$$A_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad A_{t+1} = A_t^2 - A_t - I_2, \qquad t \geq 0.$$

Where I_2 is the 2×2 identity matrix.

(a) Compute A_5, using a `do`-loop. Do not display any intermediate output.

(b) Compute and display the elements of the sequence of integers $D_t = \text{trace}(A_t)$ for $t = 1, \ldots, 5$. No other output should be displayed.

Problem 3. The *commutator* $[A, B]$ of two square matrices A and B is defined as

$$[A, B] = AB - BA.$$

(Note that if A and B were numbers, $[A, B]$ would be zero.)

(a) Construct a user-defined function `comm(A,B)` whose value is $[A, B]$.

(b) Given the matrices

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}; \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we consider the sequence of matrices

$$A_n = A^n; \qquad B_n = B^n; \qquad C_n = [A, B]^n, \qquad n \geq 1.$$

By generating a sufficient number of elements of these sequences, convince yourself that they are all periodic and hence determine their period.

Problem 4. Let

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \qquad (2)$$

(a) Construct a Maple function `mu(M)` for the function μ which subtracts 1 from all entries of a matrix M, e.g, if M is as above, then

$$\mu(M) = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}.$$

(b) Consider the recursive sequence of matrices

$$M_0 = M; \qquad M_{t+1} = M_t + \mu(M_t), \qquad t \geq 0$$

Generate the first 5 elements of this sequence.

(c) By inspecting the result of the previous problem, *conjecture* the general form of M_t, valid for $t \geq 0$.

(d) Let M be as in (2). Consider the recursive sequence of vectors of $\mathbf{Z}^3$

$$v_0 = (1, 0, 0); \qquad v_{k+1} = M\, v_k + v_0, \qquad k \geq 0.$$

Compute the smallest integer k for which the scalar product of v_k and v_0 exceeds 50.

◇ *MAPLE CHALLENGE: (for top marks)*

Problem 5. Take a four-digit integer x_0, whose digits are not all the same: ($x_0 = 1729$). Rearrange the digits to make the largest possible number (9721) and the smallest (1279). Subtract the smallest from the largest ($9721 - 1279 = 8442 = x_1$), and then repeat the above procedure on the result. We obtain a recursive sequence of four-digit integers $x_{t+1} = f(x_t)$, $t \geq 0$.

(a) Construct a user-defined function for f. [*Hint:* you may need the Maple function `sort`.]

(b) Prove with Maple that *i)* for any choice of x_0, the sequence $t \mapsto x_t$ is eventually periodic with period 1, *ii)* the repeating integer is always 6174, *iii)* the longest transient is 7.