MAS/103 Computational Mathematics I: Coursework 6

Franco VIVALDI

This coursework will be assessed and count towards your final mark for the course

DEADLINE: Wednesday of week 9, at 1:00 pm.

CONTENT: Sums and products.

PREREQUISITES: Chapter 7 of lecture notes.

MicroESSAY : Explain what is a polynomial over the rationals, and what are its degree and leading coefficient. You must use fewer than 50 words and no mathematical or Maple symbol.

Problem 1. Compute the value of the following expressions (p_k denotes the k-th prime number).

a) $$\sum_{k=1}^{100} k(k+1)$$

b) $$\sum_{k=0}^{4} \frac{k^2-10}{k^2+k+1}$$

c) $$\prod_{k=1}^{10}\left(1-\frac{1}{p_k}\right)^{-1}$$

d) $$\sum_{k=0}^{50}(-1)^k\binom{50}{k}$$

e) $$\sum_{n=0}^{5}\frac{x^n}{n!}$$

f) $$\sum_{n=1}^{5}(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}$$

g) $$\sum_{k=0}^{10}\binom{\binom{k}{2}}{2}\binom{20-k}{10}$$

h) $$\prod_{k=1}^{5}\left[\sum_{j=1}^{10}(j+k)\right]$$

i) $$\sum_{k=0}^{5}\left[\sum_{j=1}^{k+1}\frac{k^2-j}{k+j}\right]$$

j) $$\sum_{n=1}^{5}\left[\sum_{k=1}^{n}\frac{2^{k+1}}{k+1}\,p_k^n\right]$$

Problem 2. Sum and products of elements of recursive sequences.

(a) Let

$$a_0 = 2; \qquad a_{t+1} = 1 - a_t^2, \quad t \geq 0.$$

Compute

$$\sum_{t=0}^{5} a_t.$$

(b) Let

$$g_0(x) = -x; \qquad g_{t+1}(x) = \frac{x}{g_t(x)+1}, \quad t \geq 0.$$

Show that

$$\prod_{t=0}^{3} g_t(x) = \frac{x^4}{x^2 - x - 1}.$$

Problem 3. By summing a suitable characteristic function, determine

(a) The number of composite integers between 900 and 1000 (end-points included).

(b) The number of odd multiples of 31 lying between 10000 and 11000.

(c) The number of positive divisors of 16200 which are not greater than 150.

(d) The number of *integer* solutions n to the inequalities

$$n + 2 + (n+2)^3 > n^4 + 1; \qquad |n| < 10.$$

[*Hint:* interpret the rightmost inequality as a range of summation.]

(e) The number of points on the plane which have integer coordinates and which lie *inside* the circle with radius $\sqrt{10}$. How many points lie on the circumference?

◇ *MAPLE CHALLENGE: (for top marks)*

Problem 4. The Euler's totient function $\phi(n)$ counts the number of positive integers that are smaller than n and relatively prime to it. There are 4 integers smaller than 8 and relatively prime to it, namely 1,3,5 and 7. Thus $\phi(8) = 4$. Construct a user-defined function for $\phi(n)$, and hence determine an integer n such that $\phi(n)/n < 1/6$. (To speed up your computations, you may use Maple's version of ϕ, called `numtheory[phi](x)`, which is very fast.)

[*Hint:* plot the elements of the sequence $\phi(n)/n$.]