# MAS/103 Computational Mathematics I: Coursework 3

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**This coursework will be assessed and count towards your final mark for the course**

*DEADLINE: Wednesday of week 6, at 1:00 pm.*

*CONTENT: Sequences.*

*PREREQUISITES: Chapter 3 of Lecture Notes.*

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MicroESSAY : Explain what is a recursive sequence. (Use less than 50 words and no mathematical or Maple symbol whatsoever.)

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**Problem 1.** Consider the sequence $n \mapsto f_n = n/(n+1)$.

(a) Construct a function `f(n)` whose value is $f_n$.

(b) Plot the elements $f_0, \ldots, f_{30}$, connecting points with segments.

(c) Using the function `f`, generate the first 10 elements of the following sequences

$$i) \quad 0, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \ldots \qquad ii) \quad \frac{2}{3}, \frac{9}{10}, \frac{16}{17}, \frac{23}{24}, \ldots$$

$$iii) \quad \frac{1}{2^2}, \frac{3^2}{4^2}, \frac{5^2}{6^2}, \frac{7^2}{8^2}, \ldots \qquad iv) \quad 0, \frac{3}{4}, \frac{8}{9}, \frac{15}{16}, \frac{24}{25}, \ldots$$

**Problem 2.** Consider the following *recursive* sequence

$$\alpha_0 = 1; \qquad \alpha_{k+1} = f(\alpha_k) = \alpha_k^2 + 1 \qquad k \geq 0.$$

(This is the fastest-growing sequence you have come across so far.)

(a) Show that $\alpha_6$ is divisible by 1277.

(b) Show that $\alpha_9$ has 91 decimal digits. [*Hint:* how do you show that 88 has two decimal digits?]

**Problem 3.** The binomial coefficient $\binom{n}{k}$ is implemented in Maple by the function `binomial(n,k)`.

(a) Determine how many different football teams (11 players) can be made out of 30 players.

(b) Define a function `row(n)` whose value is the $n$th row of Pascal's triangle, in the form of an expression sequence. Thus

```
> row(3);
```

$$1, 3, 3, 1$$

(c) Using `row` generate the rows from 0 to 8 of Pascal's triangle. The output should look exactly as follows

$$\begin{array}{c} 1 \\ 1,\ 1 \\ 1,\ 2,\ 1 \\ 1,\ 3,\ 3,\ 1 \\ 1,\ 4,\ 6,\ 4,\ 1 \\ 1,\ 5,\ 10,\ 10,\ 5,\ 1 \\ 1,\ 6,\ 15,\ 20,\ 15,\ 6,\ 1 \\ 1,\ 7,\ 21,\ 35,\ 35,\ 21,\ 7,\ 1 \\ 1,\ 8,\ 28,\ 56,\ 70,\ 56,\ 28,\ 8,\ 1 \end{array} \tag{1}$$

Expand the binomial $(1+x)^8$ to see the binomial theorem in action, and to check your calculations.

(d) Plot the elements of the sequence

$$k \longrightarrow \binom{40}{k} \qquad k = 0, \ldots, 40$$

as isolated points.

**Problem 4.** Construct the fastest-growing sequence you can think of. Specifically, replace the 10 question marks in the following function definition

```
> fast:=n->??????????;
```

with *at most* 10 characters so as to obtain a Maple function `fast(n)`, representing a rapidly growing sequence defined for all positive $n$. (Any character that is on the keyboard is allowed, but you cannot make use of functions you have defined elsewhere.)

What counts is the asymptotic behaviour of your sequence (how fast it grows when $n$ becomes very large). For instance, $n \to n^2$ starts slower than $n \to n + 10^{10}$, but it gets to infinity first.

To get marks, your sequence must al least beat this:

```
> snail:=n->n^99999999;
```

◇ *MAPLE CHALLENGE: (for top marks)*

**Problem 5.** Let $\mathbf{Q}_I$ denote the set of rational numbers lying between 0 and 1. Consider the function

$$g : \mathbf{Q}_I \mapsto \mathbf{Q}_I \cup \{0\} \qquad\qquad g := x \mapsto \left\{\frac{1}{x}\right\}. \tag{2}$$

where $\{\cdot\}$ denotes the fractional part. Let the sequence $x_t$ be defined as follows

$$x_0 = x; \qquad\qquad x_{t+1} = \begin{cases} g(x_t) & x_t \neq 0 \\ 0 & \text{otherwise,} \end{cases} \qquad\qquad t \geq 0. \tag{3}$$

(a) Prove that for any choice of $x \in \mathbf{Q}_I$, the sequence $x_t$ with initial condition $x_0 = x$ reaches 0 in a finite number of steps.

From the above result it follows that to every rational $x$ in $Q_I$ we can associate the natural integer $\tau(x)$, which is the number of iterations needed to reach zero starting from $x$

$$\tau : \mathbf{Q}_I \mapsto \mathbf{N} \qquad\qquad \tau := x \mapsto \text{smallest } t \text{ for which } x_0 = x, \text{ and } x_t = 0.$$

For instance, $\tau(4/7) = 3$, because

$$x_0 = \frac{4}{7} \quad \to \quad x_1 = g(x_0) = \frac{3}{4} \quad \to \quad x_2 = g(x_1) = \frac{1}{3} \quad \to \quad x_3 = g(x_2) = 0. \tag{4}$$

Let $F_n$ be the subset of $\mathbf{Q}_I$ consisting of the rationals between 0 and 1, whose denominator is not greater than $n$. The set $F_n$ contains a finite number of elements. For instance $F_5$ contains nine elements

$$F_5 = \left\{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right\}.$$

Let $\Gamma_n$ be the maximum value attained by $\tau(x)$ as $x$ scans all the elements of the set $F_n$. For example, since $3/4 \in F_5$, from the data (4) we obtain the estimate $\Gamma_5 \geq 2$ (in fact $\Gamma_5 = 3$).

(b) Find a good estimate for $\Gamma_n$, in the case $n = 10000$. Specifically, obtain an estimate $\Gamma_{10000} \geq \gamma$ by producing a rational $x$ in $F_{10000}$ for which $\gamma = \tau(x)$ is a large as possible. You must display the calculations showing that the sequence with initial condition $x$ reaches zero in $\gamma$ steps. Marks will be awarded only if $\gamma \geq 10$.

[*Hint:* The set $F_{10000}$ has 30397486 elements. Careful experimentation and thinking will get you much further than blind trial and error.]