MAS/103 Computational Mathematics I: Coursework 2

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This coursework will be assessed and count towards your final mark for the course

DEADLINE: Wednesday of week 4, at 1:00 pm.

CONTENT: Prime numbers. Sets and functions.

PREREQUISITES: Sections 1.6–1.7, 2.1–2.4 of Lecture Notes

MicroESSAY : Define the integer part and the fractional part of a rational number. You may assume that such number is positive. Use fewer than 50 words and no mathematical or Maple symbols whatsoever (such as a, b, `iquo`, etc.).

Problem 1. Construct a user-defined function for the function $g(z) = z^2 - 1/z$. Hence use this function to compute the following expressions

i) $7^2 - \frac{1}{7}$ *ii)* $100 + \frac{1}{10}$ *iii)* $-a^6 + \frac{1}{a^3}$

iv) $\frac{x^2}{(x+1)^2} - \frac{x+1}{x}$ *v)* $\left(\frac{1}{y^2} - y\right)^2$ *vi)* $2\,K^2$

Problem 2. (a) Determine the number of divisors of 20!. (b) Find all divisors of 27817117.

Problem 3.

(a) There are 25 primes less or equal to 100. Indeed the 25th prime is 97, and the 26th prime is 101. Verify the above statement with the function `ithprime`.

(b) Let $\pi(x)$ be the number of primes less or equal to x. Thus $\pi(1) = 0$, $\pi(11) = 5$, $\pi(100) = 25$. Use the function `ithprime` to determine $\pi(x)$ for $x = 1000, 2000$ and 3000. (The function `nextprime` may also be useful.) This calculation is likely to require a certain amount of trial and error, but you must show only the minimal output that suffices to prove your result.

(c) Let $\Delta(x, n) = \pi(x) - \pi(x - n)$ be the number of primes that lie between $x - n$ and x ($x - n$ excluded). Tabulate the values of $\pi(x)$ and of $\Delta(x, 1000)$ (in a worksheet text region) as follows:

x	$\pi(x)$	$\Delta(x, 1000)$
1000	?	?
2000	?	?
3000	?	?

One brief comment on your findings.

Problem 4. Let $A = \{1, 2, 3\}$ and $B = \{0, 1\}$.

(a) Determine the number of distinct surjective functions $f : A \mapsto B$.

(b) Construct three distinct user-defined surjective functions $f : A \mapsto B$ [*Hint:* use `irem`, `iquo`, `abs`, etc.].

Problem 5. Construct user-defined functions for the following characteristic functions

a) $$f(x) = \begin{cases} 1 & \text{if } x \text{ is negative} \\ 0 & \text{otherwise} \end{cases}$$

Use f to decide whether or not $500^2 - 89 \cdot 53^2$ is negative.

b) $$f(x) = \begin{cases} 1 & \text{if } x \text{ is a multiple of 7} \\ 0 & \text{otherwise} \end{cases}$$

Use f to decide whether or not $100^2 + 98^2$ is divisible by 7.

c) $$f(x) = \begin{cases} 1 & \text{if } x \text{ and } x+2 \text{ are prime} \\ 0 & \text{otherwise} \end{cases}$$

Use f to decide whether or not $p_{105} - p_{104} = 2$, where p_k is the kth prime. (Think about it.)

d) $$f(x, y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are relatively prime} \\ 0 & \text{otherwise} \end{cases}$$

Use f to decide whether or not the prime 2999 divides $9! - 1$. (See example 1.7 of lecture notes.)

Problem 6. Construct the boolean characteristic function of the following sets. Use `evalb` only if necessary.

a) The set of even non-negative integers.

b) The set of rationals whose numerator is odd.

c) The set of rationals whose denominator is composite.

d) The set of primes which are twice a prime plus one.

e) The set of integers which are divisible by 5 or by 7, but not by both.

f) The set of positive integers i for which the ith prime is twice a prime plus one; hence determine the intersection between such set and the set $\{101, 102, 103, 104, 105\}$.

◇ *MAPLE CHALLENGE (for top marks)*

Problem 7. Find an integer greater than 10^{40} which has exactly 21 divisors. Such integer should be as small as possible: explain your strategy.