Queen Mary
University of London

B.Sc. EXAMINATION BY COURSE UNIT

MTH6107 Chaos & Fractals

Duration:	2 hours
Date and time:	May 02 2012, at 2.30pm
Examiner:	F Vivaldi

Attempt all questions; marks awarded are shown next to the questions.

Write your solutions in complete sentences. Marks are deducted for incorrect grammar/spelling.

- **Do not use calculators.**
- **Do not remove this paper from the exam room.**
- **Write only in answer books; cross through any work which is not to be assessed.**
- **Possession of unauthorised material —in written or electronic form— is an examination offence. If you have material you are unsure about, give it to an invigilator now.**
- **DO NOT TURN THIS PAGE UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR.**

Question 1. [Marks: (6,6,6),(4,4,4),(3,3,8),(4,7)]

(*a*)

i) Explain what is an *orbit*, a *periodic orbit*, and an *eventually periodic orbit* of a dynamical system.

ii) Explain what is a *diffeomorphism* f of $\mathbb{R}$. What are the constraints on the number and period of the periodic orbits of f? (No proof is required; consider both order-preserving and order-reversing cases.)

iii) Order the integers from 1 to 12 inclusive using Sharkovsky's ordering, then state Sharkovsky's theorem.

(*b*) Construct one-dimensional real maps with the stated properties. Justify any assertion you make.

i) A map with a single fixed point at $x = a$, where a is a given real number.

ii) A map with both bounded and unbounded orbits.

iii) A map with infinitely many periodic orbits of period 2.

(*c*)

i) Explain (without proof) how to determine the stability of a fixed point of a one-dimensional differentiable real map.

ii) Do the same for a periodic orbit of arbitrary period.

iii) Determine the (real) fixed points of the one-dimensional real map

$$f_\lambda : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \lambda - x^2$$

and analyse their stability as the real parameter λ is varied. (You may ignore the stability of the marginal cases.) What types of bifurcations take place?

(d)

i) Explain (without proof) how to determine the stability of a fixed point of a two-dimensional differentiable map $F : \mathbb{R}^2 \to \mathbb{R}^2$.

ii) Determine the fixed points of the two-dimensional map

$$F_\lambda : \mathbb{R}^2 \to \mathbb{R}^2, \qquad F(x,y) \equiv (1 - \lambda^2 x^2 + y,\, x)$$

and analyse their stability as the real parameter λ is varied.

Question 2. [Marks: 4,5]

(a) Define the *Lyapouvov exponent* of a differentiable real map at a point.

(b) Let λ be a real number. Determine the Lyapounov exponent of the real map $x \mapsto \lambda x$, and show that it is the same at every point.

Question 3. [Marks: 4,7] Consider the doubling map

$$f : [0,1) \to [0,1), \qquad f(x) \equiv 2x \,(\text{mod } 1).$$

(a) Prove that all periodic orbits of this map are unstable.

(b) Determine one periodic orbit of period 3. How many periodic orbits of period 4 does this map have?

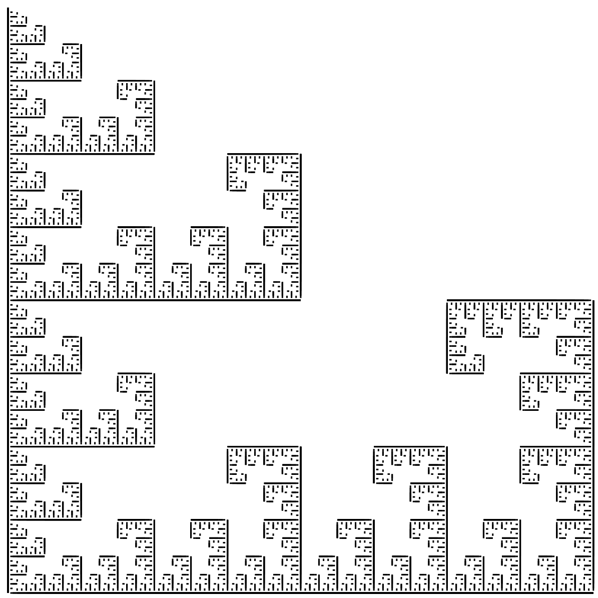
Question 4. [Marks: (5,5),(5,5,5)]

(*a*)

i) Define the *Hausdorff distance* between two subsets of the plane.

ii) Define the *box dimension* of a subset of the plane, and explain what is a *fractal*.

(*b*) Let A be the following set



i) After choosing the size of A, compute the Hausdorff distance between A and the smallest filled-in square containing A.

ii) After choosing co-ordinates, construct an iterated function system whose fixed point is A.

iii) Compute the box dimension of A.

End of examination paper