METRIC SPACES

A <u>metric space</u> is a set X, together with a <u>distance</u> d, namely a non-negative real function $d: X \times X \to \mathbb{R}$ with the following properties, valid for all $x, y, z \in X$

1) $d(x,y) = 0$ if and only if $x = y$
2) $d(x,y) = d(y,x)$ (symmetry)
3) $d(x,z) \leq d(x,y) + d(y,z)$. (triangle inequality)

One may define different distances on the same set, leading to different metric spaces. For example, let $X = \mathbb{R}^n$, with elements $x = (x_1, \dots, x_n)$, $x_k \in \mathbb{R}$. We define various distances on $\mathbb{R}^n$

$$d(x,y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2} \qquad \text{(Euclidean)}$$

$$d_1(x,y) = \sum_{k=1}^{n} |x_k - y_k|$$

$$d_0(x,y) = \max_{1 \leq k \leq n} |x_k - y_k|.$$

One verifies that d, d_1, d_0 are distance functions.

In what follows, X is a metric space with a distance d, x is a point of X, and $A \subset X$.

An <u>open sphere</u> (or open ball) $S(x_0, r)$ is the set $\{x \in X : d(x, x_0) < r\}$. The point x_0 is called the <u>centre</u> of the sphere and the number r its <u>radius</u>.

A point x is an <u>interior point</u> of a set A if there is an open sphere $S(x, r)$ contained in A. A set is <u>open</u> if all its points are interior points.

A <u>neighbourhood</u> of x is an open set containing x. An <u>ε-neighbourhood</u> of x is a neighbourhood of x contained in $S(x, \varepsilon)$.

A point x is a <u>contact point</u> of a set A if every neighbourhood of x contains at least one point of A.

The <u>closure</u> of A, denoted by $\overline{A}$, is the set of all contact points of A. A set is <u>closed</u> if it coincides with its own closure.

A point x is a <u>limit point</u> (or <u>accomolation point</u>) of a set A if every neighbourhood of x contains infinitely many points of A.

A point x is <u>isolated</u> if there is a ("sufficientely small") neighbourhood of x that contains no other point of A

A point x is a <u>boundary point</u> of A if every neighbourhood of x contains a point of A as well as a point of its complement $X \setminus A$.

A set A is <u>dense</u> in a set B if $\bar{A} \supset B$.
A is everywhere dense if $\bar{A} = \bar{X}$, and nowhere dense if it is dense in no open sphere.

Every interior point is a contact point, and a limit point. Every isolated point is a contact point and a boundary point, but not a limit point. Every limit point is a contact point.

SOME BASIC THEOREMS

Thm 1 The closure operator has the following properties

1) If $A \subset B$, then $\overline{A} \subset \overline{B}$
2) $\overline{\overline{A}} = \overline{A}$
3) $\overline{A \cup B} = \overline{A} \cup \overline{B}$
4) $\overline{\emptyset} = \emptyset$

Thm 2 A point x is a contact point of A iff there exists a sequence of points of A converging to x.
A point x is a limit point of A iff there exists a sequence of distinct points of A converging to x

Thm 3 A subset A of a metric space X is open iff its complement $X \setminus A$ is closed. In particular, the space X and the empty set are both open and closed.

Thm 4 The intersection of an arbitrary number of closed set is closed. The union of a finite number of closed sets is closed.
The union of an arbitrary number of open sets is open. The intersection of a finite number of open sets is open.

Thm 5 Every open set on the real line is the union of a finite or countable set of pairwise disjoint open intervals.

EXAMPLES

Let $X = \mathbb{R}$, with the Euclidean metric.

Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Every point of A is isolated, and $\bar{A} = A \cup \{0\}$. The point $x = 0$ is a limit point of A, which does not belong to A. The set A is nowhere dense.

The set $\mathbb{Q}$ is everywhere dense in $\mathbb{R}$. All points of $\mathbb{Q}$ are boundary points, that is, $\mathbb{Q}$ has empty interior. All points of $\mathbb{Q}$ are limit points.

The set $\mathbb{Z}$ is nowhere dense; all its points are isolated, and it has no accomulation point.

Every open interval is open; every closed interval is closed, and so is every single element set. (Thm 3,5)

The union of an infinite number of closed intervals is not necessarily closed (cf Thm 4)

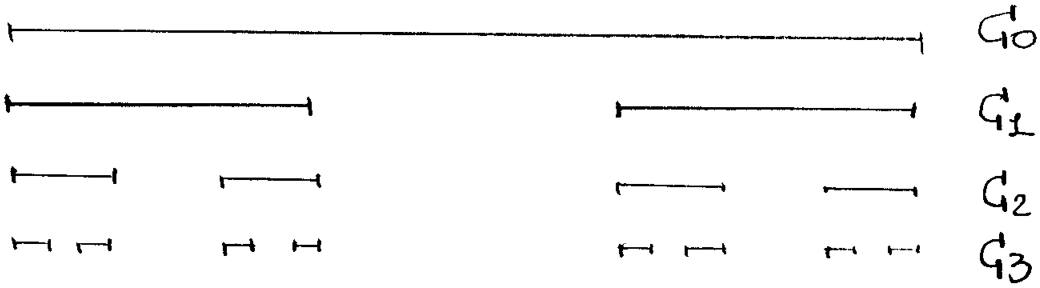
$$[0,1) = \bigcup_{n=1}^{\infty} [0, 1-\tfrac{1}{n}]$$

The intersection of an infinite number of open intervals is not necessarily open

$$(0,1] = \bigcap_{n=1}^{\infty} (0, 1+\tfrac{1}{n})$$

The Cantor ternary set

This is a subset $\mathcal{C}$ of the closed unit interval, constructed recursively as follows. We start from the closed unit interval G_0 we divide it into three equal sub-intervals, and we remove the open middle interval $(1/3, 2/3)$, to obtain a new set G_1. We repeat the same procedure to each of the two closed sub-intervals comprising G_1 to obtain a new set G_2 and so forth, ad infinitum



The limit set

$$\mathcal{C} := \bigcap_{k=0}^{\infty} G_k$$

is closed, from theorem 4. A point $x \in \mathcal{C}$ can be represented as a series

$$x = \sum_{k=1}^{\infty} d_k \frac{1}{3^k} \qquad d_k \in \{0, 2\}. \qquad (1)$$

This is an expansion in base 3, without the digit 1. The missing digit $d_k = 1$ accounts for the deletion of all open middle intervals at the kth stage of the recursion.

With x and d_k as in (1), the mapping

$$x \mapsto \frac{1}{2}\sum_{k=1}^{\infty} d_k \frac{1}{2^k} = \sum_{k=1}^{\infty} d'_k \frac{1}{2^k} \qquad d'_k = \frac{d_k}{2} \in \{0, 1\}$$

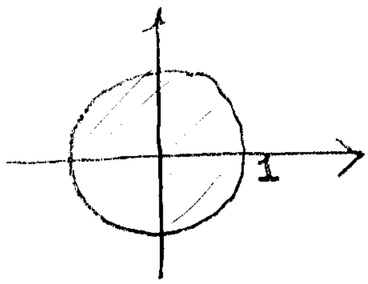
defines a surjection from $\mathcal{C}$ onto the whole interval $[0,1]$. Such a map is invertible, except at the rational points of the type $\frac{n}{2^m}$, which admit two distinct binary representations (terminating with all 0 and all 1, respectively).

All points of $\mathcal{C}$ are accumulation points (this follows from (1)), although $\mathcal{C}$ has empty interior. To see this, we note that the total length of the intervals comprising G_k is $\left(\frac{2}{3}\right)^k \xrightarrow[k\to\infty]{} 0$, and hence $\mathcal{C}$ cannot contain any interval.

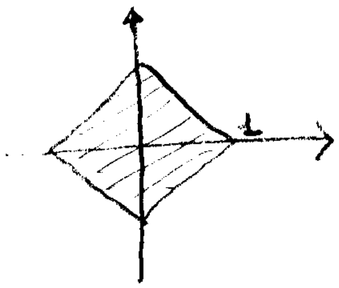
All expansion (1) whose digits are eventually constant (either 0, or 2), are rational points in $\mathcal{C}$. They are the end-points of the segments comprising G_k. These rational points are <u>dense</u> in $\mathcal{C}$. The cantor set $\mathcal{C}$ contains infinitely many other rational points, corresponding to digit sequences that are eventually periodic of period greater than one.

Let $X = \mathbb{R}^2$

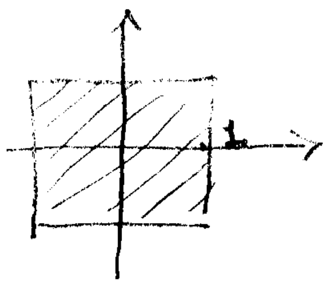
Ball $S(0, 1)$



distance d



distance d_1



distance d_0