Chaos and fractals

Exercises and solutions

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Chapter 1

Asymptotics

Let f and g be real functions. We write $f(x) \prec g(x)$ to mean that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0 \tag{1}$$

and $f(x) \sim g(x)$ to mean that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1. \tag{2}$$

Exercise 1.

(a) Prove that if $f(x) \prec g(x)$, then $f(x) = O(g(x))$, as $x \to \infty$.
[*The big-O definition requires a constant; find it using the definition of the limit (1).*]

(b) Prove that the converse is not true, by giving a counterexample.
[*Is* $f(x) = O(f(x))$*?*]

Exercise 2. Prove that if $g(x) \to \infty$, then

$$f(x) \prec g(x) \quad \Rightarrow \quad e^{f(x)} \prec e^{g(x)}.$$

Is the converse implication true?

Exercise 3. Let ε and c be real numbers with $0 < \varepsilon < 1 < c$. Order the following functions according to the $\prec$ relation

$$x^c, \quad c^x, \quad c^{c^x}, \quad x^\varepsilon, \quad \varepsilon^x, \quad 1, \quad x^{\log(x)}, \quad \log(x), \quad \log(\log(x)), \quad \log(x)^c.$$

[*Use Hôpital's rule; take logarithms and then use problem 2.*]

Exercise 4. Which function grows faster

$$x^{\log(x)} \qquad \text{or} \qquad \log(x)^x?$$

[*How are these functions defined? Use problem 2.*]

Exercise 5. Find an asymptotic expression for the function

$$f(x) = \frac{x + \log(xe^x)}{1 + \log(x^3)} + \frac{x}{\log(x)\sqrt{\log(x)}}$$

as $x \to \infty$. Namely, find a function g simpler than f, such that $f \sim g$.
[*Identify the dominant term of each numerator and denominator, than that of each fraction, etc. (What's the logarithm of a product?)*]

Exercise 6. Prove that

$$(1 + 2x + O(x^2)) = (1 + 2x)(1 + O(x^2)) \qquad x \to 0.$$

Exercise 7. Prove or disprove

1. $\dfrac{1}{1+x^2} = 1 + O(x)$ $\qquad x \to 0$
2. $\cos(x)\sin(x) = O(x^2)$ $\qquad x \to \infty$
3. $\cos(x)\sin(x) = O(x^2)$ $\qquad x \to 0$
4. $\cos(O(x)) = 1 + O(x^2)$ $\qquad$ all x
5. $O(x+y) = O(x^2) + O(y^2)$ $\qquad x, y \to \infty$
6. $e^{(1+O(1/n))^2} = e + O(1/n)$ $\qquad n \to \infty$
7. $n^{\log(n)} = O(\log(n)^n)$ $\qquad n \to \infty$

Exercise 8. Multiply $(\log(n)+\gamma+O(1/n))$ by $(n+O(\sqrt{n}))$ and express your answer in O-notation.

Solutions

Solution 1. Let

(a)
$$\lim_{x\to\infty}\frac{f(x)}{g(x)}\to 0.$$
Fix an arbitrary $\varepsilon>0$. By definition of limit, there is a real constant C, such that, for all $x>C$, we have
$$\left|\frac{f(x)}{(x)}\right|<\varepsilon.$$
This means that, for all sufficiently large x, we have
$$|f(x)|<\varepsilon|g(x)|,$$
which states that $f(x)=O(g(x))$, as desired.

(b) We show that the converse statement is false. Indeed, for all functions f we have $f(x)=O(f(x))$. However,
$$\lim_{x\to\infty}\frac{f(x)}{f(x)}=1\neq 0,$$
and hence $f(x)\not\prec f(x)$.

Solution 2. Assume that $f(x)\prec g(x)$, and that g tends to infinity. We must show that
$$\frac{e^{f(x)}}{e^{g(x)}}=e^{f(x)-g(x)}\to 0$$
that is, that $f(x)-g(x)\to-\infty$.

We have
$$f(x)-g(x)=\frac{f(x)-g(x)}{g(x)}g(x)=\left(\frac{f(x)}{g(x)}-1\right)g(x)\sim -g(x)\to-\infty.$$
Alternatively, choose $\varepsilon<1$. Then for all sufficiently large x, we have $f(x)\leq |f(x)|<\varepsilon|g(x)|=\varepsilon g(x)$ (g is positive), or $f(x)-g(x)<g(x)(1-\varepsilon)\to-\infty$.

The converse implication is false. We have $e^x\prec e^{2x}$, but $x\not\prec 2x$.

Solution 3. We find

$$\varepsilon^x \prec 1 \prec \log(\log(x)) \prec \log(x) \prec \log(x)^c \prec x^\varepsilon \prec x^c \prec x^{\log(x)} \prec c^x \prec x^x \prec c^{c^x}.$$

The only tricky relation is $x^x \prec c^{c^x}$, which is established by taking logarithms, and using problem 2.

Solution 4. We will show, in three different ways, that

$$x^{\log(x)} \prec \log(x)^x.$$

By definition, we have

$$x^{\log(x)} = e^{\log(x)^2} \qquad \log(x)^x = e^{x\log(\log(x))}.$$

We compare the exponents

$$\log(x)^2 \qquad \text{and} \qquad x\log(\log(x)).$$

From the previous problem, we have $\log(x)^2 \prec x$, and hence, $\log(x)^2 \prec x\log(\log(x))$. The desired result now follows from problem 2.

Alternatively, let

$$f(x) = \frac{x^{\log(x)}}{\log(x)^x}.$$

We take the logarithm of both sides, and then divide by $\log(x)$, to obtain

$$g(x) = \frac{\log(f(x))}{\log(x)} = \frac{\log(x)^2 - x\log(x)}{\log(x)} = \log(x) - x.$$

It is now clear that

$$\lim_{x\to\infty} g(x) = -\infty = \lim_{x\to\infty} \log(f(x))$$

and hence

$$\lim_{x\to\infty} \frac{x^{\log(x)}}{\log(x)^x} =, \lim_{x\to\infty} f(x) = 0,$$

as desired.

Finally, for every constant $c > 1$, we clearly have

$$c^x \prec \log(x)^x. \tag{2}$$

Furthermore, we have

$$\log(x)^2 \prec x$$

and hence

$$\log(x)^2 \prec x\log(c).$$

Taking exponentials, we find

$$x^{\log(x)^2} \prec c^x,$$

which, together with (2), gives the desired result.

Solution 5. We find

$$\begin{aligned} f(x) &= \frac{2x+\log(x)}{1+3\log(x)} + \frac{x}{\log(x)^{3/2}} \\ &= \frac{2x}{3\log(x)}\frac{1+\log(x)/2x}{1+1/(3\log(x))} + \frac{x}{\log(x)^{3/2}} \\ &= \frac{2x}{3\log(x)}\left[\frac{1+\log(x)/2x}{1+1/(3\log(x))} + \frac{3}{2\sqrt{\log x}}\right]. \end{aligned}$$

As $x \to \infty$, the quantity within square brackets tends to 1. Thus

$$f(x) \sim \frac{2x}{3\log(x)}.$$

Solution 6. We expand the product

$$(1+2x)(1+O(x^2)) = 1+2x+O(x^2)+2xO(x^2) = 1+2x+O(x^2)+O(x^3).$$

As $x \to 0$, we have $O(x^2)+O(x^3) = O(x^2)$, and the result follows.

Solution 7.

1. TRUE. Since $x \to 0$, we expand in Taylor series near zero

$$\frac{1}{1+x^2} = 1+x^2+x^4+\cdots = 1+O(x^2) = 1+O(x).$$

2. TRUE. Sine and cosine are *bounded* functions, and so is their product. So we may write $\sin(x)\cos(x) = O(1)$, for all x. Now $O(1) = O(x^2)$, as $x \to \infty$.

3. FALSE. Expanding in Taylor series near 0, we have

$$\begin{aligned}\sin(x)\cos(x) &= (x+O(x^3))(1+O(x^2))\\ &= x+O(x^3)+xO(x^2)+O(x^3)O(x^2)\\ &= x+O(x^3)+O(x^3)+O(x^5)=x+O(x^3).\end{aligned}$$

   The identity $x+O(x^3)=O(x^2)$ is false.

4. TRUE. As $x \to 0$, the statement follows from the Taylor expansion

$$\cos(O(x)) = 1+O(x)^2 = 1+O(x^2).$$

   As $x \to \infty$, we have $\cos(x) = O(1) = O(x^2)$.

   This proof is a bit sloppy. We now develop a rigorous argument, from the definition of O. We have to show that the set L on the left is a subset of the set R on the right. If a function f belongs to L, then we can write $f(x) = \cos(g(x))$, where $|g(x)| < c|x|$, for some c. Then

$$1-f(x) = 2\sin(g(x)/2)^2 \le \frac{1}{2}g(x)^2 \le \frac{1}{2}c^2x^2.$$

   This shows that $1-f(x) \in O(x^2)$, that is, that $f(x) \in 1+O(x^2) = R$.

   Thus $f \in L \Rightarrow f \in R$, that is, $L \subset R$, as desired.

5. TRUE. We have

$$\begin{aligned}O(x+y)^2 &= O((x+y)^2) = O(x^2+y^2+2xy) = O(O(x^2)+O(y^2))\\ &= O(O(x^2))+O(O(y^2)) = O(x^2)+O(y^2).\end{aligned}$$

6. TRUE. First, we take care of the exponent. As $n \to \infty$, we find

$$(1+O(1/n))^2 = 1+2O(1/n)+O(1/n)^2 = 1+O(1/n).$$

   Second, we expand the exponential

$$e^{1+O(1/n)} = ee^{O(1/n)} = e(1+O(1/n)) = e+eO(1/n) = e+O(1/n).$$

7. TRUE. This follows from problem 1(a) and problem 4.

Solution 8. We expand the product

$$\begin{aligned}(\log(n)+\gamma+O(1/n))(n+O(\sqrt{n})) &= n\log(n)+\gamma n+O(1)+O(\sqrt{n}\log(n))\\ &\quad +O(\sqrt{n})+O(1/\sqrt{n}).\end{aligned}$$

As $n\to\infty$, the dominant term is $O(\sqrt{n}\log(n))$, namely

$$O(1/n)+O(1)+O(\sqrt{n})+O(\sqrt{n}\log(n)) = O(\sqrt{n}\log(n)).$$

So we find

$$(\log(n)+\gamma+O(1/n))(n+O(\sqrt{n})) = n\log(n)+\gamma n+O(\sqrt{n}\log(n)).$$

Chapter 2

Periodic points

Exercise 1. Let the phase space Σ be the set $\mathbb{N}$ of positive integers, and let the dynamics $f : \mathbb{N} \to \mathbb{N}$ be given by

$$f(x) = \begin{cases} x/2 & x \text{ even} \\ 3x+1 & x \text{ odd.} \end{cases}$$

(a) Prove that f has no fixed points.

(b) Show that all points $x \leq 10$ are (eventually) periodic, and compute their transient length, i.e., the length of the pre-periodic part of the orbit through x. (With the help of a numerical experiment, you should be able to formulate the '$3x+1$ conjecture', one of the most famous unresolved puzzles in number theory.)

Exercise 2. We represent a real number x as the sum of its integer and fractional parts, namely

$$x = \lfloor x \rfloor + \{x\}.$$

The integer part $\lfloor x \rfloor$ (also called the *floor* of x) is the largest integer not exceeding x, while the fractional part has values in the range $0 \leq \{x\} < 1$.

Now let $\Sigma = [0, 1]$ and

$$f(x) = \begin{cases} \{1/x\} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Prove that f has infinitely many fixed points, and determine all of them.
[*The tricky bit is to plot the graph of the function* $x \mapsto \{1/x\}$. *Begin with the restricted domain* $1/3 < x < 1$.]

Exercise 3. Let $\Sigma = [0,1]$ and $f(x) = 4x(1-x)$.

(a) Determine all fixed points.

(b) For each fixed point, determine three eventually fixed points.
[*Construct pre-images of fixed points under f, and then their pre-images*.]

(c) Determine all 2-cycles.
[*Let $P_k(x) = f^k(x) - x$. The 2-cycles which are not 1-cycles are the roots of the polynomial $P_2(x)/P_1(x)$. (Why is P_2/P_1 a polynomial? Think about it.)*]

Exercise 4. Let $\Sigma = [0,1]$, and let f be the *tent map*, defined as

$$f(x) = \begin{cases} 2x & 0 \leqslant x < 1/2 \\ 2-2x & 1/2 \leqslant x \leqslant 1. \end{cases}$$

(a) Construct the function f^2.

(b) The point $x = 0$ is a fixed point. Characterise the set Θ all the points in Σ that eventually reach $x = 0$.

Exercise 5*. Let $\mathbb{Z}[x]$ be the set of polynomials with integer coefficients, and let $f : \mathbb{Z}[x] \to \mathbb{Z}[x]$ be defined by the formula

$$p(x) \mapsto x^n p(1/x) \qquad n = \deg(p)$$

where $\deg(p)$ is the degree of p. Prove that all points of $\mathbb{Z}[x]$ are periodic with period dividing 2. Characterise the fixed points.

Solutions

Solution 1.

(a) If x is even, then $f(x) < x$; if x is odd, then $f(x) > x$. So we cannot have $f(x) = x$.

(b) We compute each orbit, until it reaches a point already computed.

orbit	transient length
$[1, 4, 2, 1, \ldots]$ (a 3-cycle)	0
$[2, \ldots]$	0
$[3, 10, 5, 16, 8, 4, \ldots]$	5
$[4, \ldots]$	0
$[5, \ldots]$	3
$[6, 3, \ldots]$	6
$[7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, \ldots]$	14
$[8, \ldots]$	1
$[9, 28, 14, 7, \ldots]$	17
$[10, \ldots]$	4

So all points x such that $1 \leq x \leq 10$ are (eventually) periodic with period 3. The '$3x + 1$ conjecture' states that all orbits end up in the above 3-cycle.

Solution 2. The point $x = 0$ is a fixed point. For $x \neq 0$, the fixed point equation $x = f(x)$ reads

$$\left\{\frac{1}{x}\right\} = x \qquad \Longleftrightarrow \qquad \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = x.$$

Let m be the floor of $1/x$. We write the fixed-point equation explicitly:

$$\frac{1}{x} - m = x, \qquad \text{where} \qquad \frac{1}{m+1} \leq x < \frac{1}{m}, \quad m = 1, 2, \ldots.$$

This gives $x^2 + mx - 1 = 0$, and hence

$$x = \frac{-m \pm \sqrt{m^2 + 4}}{2}.$$

The negative sign does not yield points of Σ. One verifies that (do it!)

$$0 < \frac{-m + \sqrt{m^2 + 4}}{2} < 1 \qquad m = 1, 2, \ldots,$$

giving infinitely many fixed points.

Solution 3.

(a) The fixed-point equation yields

$$P_1(x) = x(4x-3) = 0,$$

giving two fixed points: $x^* = 0$ and $x^* = 3/4$.

(b) Letting $y = f(x)$, we find

$$x = f^{-1}(y) = \frac{1 \pm \sqrt{1-y}}{2}.$$

Using f^{-1}, we compute pre-images of fixed points. For $x^* = 0$ we find

$$x^* = 0 \leftarrow \begin{cases} 0 = x^* \\ 1 \leftarrow \frac{1}{2} \quad \leftarrow \begin{cases} \frac{2+\sqrt{2}}{4} \\ \frac{2-\sqrt{2}}{4} \end{cases} \end{cases}$$

Note that the point 1 has a single pre-image, which is the critical point of the function f. For $x^* = 3/4$ we obtain

$$x^* = \frac{3}{4} \leftarrow \begin{cases} \frac{3}{4} = x^* \\ \frac{1}{4} \quad \leftarrow \begin{cases} \frac{2+\sqrt{3}}{4} \\ \frac{2-\sqrt{3}}{4} \end{cases} \end{cases}$$

(c) For period 2 we we must solve the equation $f^2(x) = x$, which yields

$$P_2(x) = 64x^4 - 128x^3 + 80x^2 - 15x = 0.$$

We eliminate the fixed points. Long division gives

$$\frac{P_2(x)}{P_1(x)} = 16x^2 - 20x + 5$$

and this polynomial has roots

$$x^*_\pm = \frac{5 \pm \sqrt{5}}{8},$$

which constitute the desired 2-cycle, since they are distinct (they need not be).

Solution 4.

(a) Since f is a piecewise-defined, so is f^2. It has four pieces. (Why?)

(b) We have

$$f^{-1}(\{0\}) = \{0,1\} \qquad f^{-1}(\{1\}) = \{1/2\}.$$

Let $\Omega \subset \Sigma$ be the set of all rational points in the unit interval whose denominator is a non-negative power of 2. We claim that $\Omega = \Theta$, the set of all pre-images of the origin.

We proceed by induction on the power n of 2 at denominator. The points $0, 1, 1/2$ are in Ω, which is the base case. Assume now that for some $n \geqslant 1$, all rational of the form $x = m/2^n$ are pre-images of the origin, where m is an odd integer in the range $0 < m < 2^n$. Then we have

$$f^{-1}(\{m/2^n\}) = \{m/2^{n+1}, 1 - m/2^{n+1}\} = \{m/2^{n+1}, (2^{n+1} - m)/2^{n+1}\} \subset \Omega.$$

This completes the inductive step, and we have shown that $\Theta \subset \Omega$.

Conversely, let $x = m/2^n \in \Omega$ be given. If $x < 1/2$ then $f(x) = m/2^{n-1}$, and if $x \geqslant 1/2$ then $f(x) = (2^{n-1} - m)/2^{n-1}$. Furthermore in each case the power of 2 decreases by 1. So $f^n(x) = 1$, and x is a pre-image of the origin. This shows that $\Omega \subset \Theta$, and hence $\Omega = \Theta$.

Hint 5. Experiment with quadratic polynomials.

Chapter 3

Stability of periodic orbits

Exercise 1. Explain concisely what is a superstable periodic orbit. Try not to use symbols.

Exercise 2. Let $f_\lambda(x) = x + \lambda \sin(x)$, with $\lambda > 0$.

(a) Show that this map has infinitely many fixed points.

(b) Determine the range of values of λ for which f has stable fixed points. (They are subdivided into two families.) For what value of λ are they superstable?

(c) Determine the nature of the fixed points at $\lambda = 2$.

Exercise 3. Let $\Sigma = [0,1]$ and $f_\mu(x) = \mu x(1-x)$ with $\mu \geq 1$.

(a) Show that f has a superstable periodic orbit if such an orbit contains the point $x = 1/2$.

(b) Show that at $\mu = 1$, the map has a single fixed point, and two fixed points for $\mu > 1$.

(c) Determine the μ-range for which f has a stable 1-cycle. When is such a cycle superstable?

(d) Determine the nature of the fixed point at $\mu = 3$. [*Hint:* look at f^2, near the fixed point.]

(e) Determine the μ-range for which f has a stable 2-cycle. When is such a cycle superstable?

Exercise 4. Let x^* be such that $g(x^*) = 0$, where g is a differentiable real function.

(a) Show that if $g'(x^*) \neq 0$, then x^* is a superstable fixed point for Newton's method.

(b) Let $g'(x^*) = 0$, and let $k > 1$ be the smallest integer for which

$$\frac{d^k g(x^*)}{dx^k} \neq 0.$$

Show that x^* is an attractor, but not a superstable one. Determine the multiplier at x^*, as a function of k. Comment on your findings. [*Hint:* expand the multiplier in Taylor series.]

Solutions

Solution 1. A periodic orbit is said to be *superstable* if its multiplier is equal to zero, that is, if one point of the orbit is a critical point. The convergence to a superstable orbit is faster than exponential. This means that the distance of a point from the cycle is of the order of the square of the distance of the pre-image(s) of this point.

Solution 2.

(a) The fixed point equation $\lambda \sin(x) = 0$ gives infinitely many fixed points: $x_k^* = k\pi$, $k \in \mathbb{Z}$, independent of λ.

(b) The multiplier is $f'(x) = 1 + \lambda \cos(x)$, so that

$$f'(x_k^*) = 1 + \lambda \cos(k\pi) = \begin{cases} 1+\lambda & k \text{ even (type I)} \\ 1-\lambda & k \text{ odd (type II).} \end{cases}$$

Type I fixed points are always unstable, since λ is positive. Type II fixed points are stable provided that $0 < \lambda < 2$, and superstable for $\lambda = 1$.

(a) For $\lambda = 2$, linear analysis fails to establish the nature of type II fixed points, so we need to include nonlinear terms. We expand $f(x)$ near $x = \pi$ in Taylor series, letting $x = \pi + \delta$. We find

$$f'(x) = 1 + 2\cos(x) \qquad f''(x) = -2\sin(x) \qquad f'''(x) = -2\cos(x)$$

giving

$$f'(\pi) = -1 \qquad f''(\pi) = 0 \qquad f'''(\pi) = 2$$

so that

$$f(\pi+\delta) = \pi - \delta + \frac{1}{3}\delta^3 + O(\delta^5).$$

Thus

$$\delta_{t+1} = \delta_t\left(-1 + \frac{1}{3}\delta_t^2 + O(\delta_t^4)\right).$$

For sufficiently small $|\delta_t|$, the quantity in parenthesis is smaller that 1 in absolute value. So δ decreases, and these points are attractors.

Solution 3.

(a) We have $f'(x) = \mu(1-2x)$, so there is a unique critical point $x_c = 1/2$, which must belong to any superstable cycle.

(b) The map f has two fixed points, namely 0 and $x^*(\mu) = 1-\mu^{-1}$. For $\mu = 1$ they coincide.

(c) For $\mu > 0$, we have $f'(0) > 1$, so 0 is unstable, whereas $f'(x^*) = 2-\mu$. This means that x^* is stable in the range $1 < \mu < 3$, and it is superstable for $\mu = 2$.

(d) When $\mu = 3$, we have $x^* = 2/3$. Let $x = x^* + \delta$. We compute

$$g(\delta) = f(\delta) - x^* = 3(2/3+\delta)(1-2/3-\delta) = -\delta - 3\,\delta^2.$$

Then

$$g^2(\delta) = \delta - 18\delta^3 + 27\delta^4 = \delta\,(1 - 18\,\delta^2 + O(\delta^3)).$$

Thus, for sufficiently small $|\delta|$, the quantity in parenthesis is smaller than 1, and x^* is an attractor.

(e) The 2-cycles are the roots of the polynomial

$$\phi_2(x) = \frac{f^2(x)-x}{f(x)-x} = \mu^2x^2 - (\mu^2+\mu)\,x + \mu + 1.$$

Let x_1 and x_2 be the roots of $\phi_2(x)$. Then, looking at the coefficients of $\phi_2(x)$, we deduce that

$$x_1 + x_2 = \frac{\mu^2+\mu}{\mu^2} = \frac{\mu+1}{\mu} \qquad\qquad x_1\,x_2 = \frac{\mu+1}{\mu^2}.$$

So the multiplier of the 2-cycle is given by

$$\omega(\mu) = f'(x_1)\, f'(x_2) = \mu^2(1-2(x_1+x_2)+4x_1x_2) = -\mu^2+2\mu+4.$$

The 2-cycle is born at $\omega(\mu)=1$, or

$$\omega(\mu)-1 = -(\mu+1)(\mu-3) = 0$$

giving $\mu=3$, as desired (the other root is out of range). The 2-cycle is superstable if $\omega(\mu)=0$, that is, if $\mu = 1+\sqrt{5}$ (we have chosen the positive sign in front of the radical). This cycle bifurcates to a 4-cycle if $\omega(\mu)=-1$, that is, if $\mu = 1+\sqrt{6}$.

Solution 4.

(a) Let $f(x) = x - g(x)/g'(x)$. Then

$$f'(x) = 1 - \frac{g'(x)^2 - g(x)g''(x)}{g'(x)^2} = \frac{g(x)g''(x)}{g'(x)^2}.$$

Since, by assumption, $g(x^*)=0$ and $g'(x^*)\neq 0$, then $f'(x^*)=0$, that is, x^* is superstable.

(b) Let $\delta = x - x^*$, and let $a = d^kg/dx^k(x^*)$. From Taylor's theorem we have

$$g(x) = \frac{a}{k!}\delta^k + O(\delta^{k+1}),$$

whence

$$g'(x) = \frac{a}{(k-1)!}\delta^{k-1} + O(\delta^k) \qquad\qquad g''(x) = \frac{a}{(k-2)!}\delta^{k-2} + O(\delta^{k-1}).$$

From the above we obtain

$$\begin{aligned} f'(x) &= f'(x^*+\delta) = \frac{g(x)g''(x)}{g'(x)^2} = \frac{\dfrac{a^2}{k!\,(k-2)!}\,\delta^{2k-2} + O(\delta^{2k})}{\dfrac{a^2}{(k-1)!^2}\,\delta^{2k-2} + O(\delta^{2k})} \\ &= \frac{\dfrac{a^2}{k!\,(k-2)!} + O(\delta^2)}{\dfrac{a^2}{(k-1)!^2} + O(\delta^2)}. \end{aligned}$$

The value at $x = x^*$ is obtained by letting δ tend to zero:

$$f'(x^*) = \lim_{\delta \to 0} f'(x^* + \delta) = \frac{(k-1)!^2}{k!\,(k-2)!} = 1 - \frac{1}{k}.$$

The convergence is exponential rather than super-exponential, with rate $1 - k^{-1}$. Thus the convergence is maximal when $k = 2$, that is, when the second derivative of g does not vanish at the root x^*.

Chapter 4

Renormalisation

Exercise 1. Explain what is a bifurcation tree. Try not to use symbols.

Exercise 2. Describe the Feigenbaum-Cvitanović equation, and the significance of its solutions. Try not to use symbols.

Exercise 3. Consider the map $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} x + 2x^2 & 0 \leq x < \frac{1}{2} \\ \\ 2 - 2x & \frac{1}{2} \leq x \leq 1. \end{cases}$$

(a) Prove that f has no stable fixed points.

(b) Prove that f has no stable periodic points of any period.
[*Look at the multiplier, using the chain rule of differentiation. Consider the point* $x = 1/2$ *separately.*]

(c) Find a 2-cycle and compute its multiplier.

Exercise 4. Let f be as in the previous problem.

(a) Show that in the vicinity of 0, the function f is an *approximate* solution to the Feigenbaum-Cvitanović equation, in the sense that

$$\alpha f\left(f(x/\alpha)\right) = f(x) + O(x^3).$$

What value do you get for the scaling α in this case?

(b) Improve the accuracy of the above approximation as follows. Let $\hat{f}(x) = f(x) + cx^3$, and determine the parameter c so that

$$\alpha \hat{f}\left(\hat{f}(x/\alpha)\right) = \hat{f}(x) + O(x^4).$$

(c) Let $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = \frac{x}{1 - bx}$$

where b is a real constant. Show that this function is an *exact* solution to the Feigenbaum-Cvitanović equation. (This solution is not relevant to period-doubling, since it has the normalisation $g(0) = 0$. It is relevant to intermittency.) Relate the function g to the functions f and $\hat{f}$ above.
[*Expand* $g(x)$ *in Taylor series about zero.*]

Exercise 5. Consider the logistic map $f_\lambda(x) = 1 - \lambda x^2$. Prove that if $\lambda > 2$, some orbits from the interval $[-1, 1]$ escape to infinity.
[*Look at the orbit of the critical point. First draw a picture, then from it construct a rigorous argument.*]

Exercise 6. Let Σ be the space of real analytic functions $f : \mathbb{R} \to \mathbb{R}$, and let $D : \Sigma \to \Sigma$ be the differentiation operator

$$(D\,f)(x) = \frac{d\,f(x)}{dx}.$$

Show that D has a one-parameter family of fixed points. Find a 2-cycle and a 4-cycle. Can you find eventually periodic points?

Exercise 7. Let Σ be the space of real functions f analytic at $x = 0$ and normalised so that $f(0) = 0$. Let R be the operator defined as

$$(Rf)(x) = \alpha f(x/\alpha) \qquad\qquad \alpha > 1.$$

Prove that $R : \Sigma \to \Sigma$. Specifically, prove that $(Rf)(0) = 0$, and that if ρ is the radius of convergence of f, then the radius of convergence of Rf is $\alpha\rho$. What happens to the radius of convergence under repeated iterations of R?

Exercise 8. Consider the Feigenbaum-Cvitanović equation for period-doubling

$$(Rf)(x) = \alpha f(f(x/\alpha)) \qquad f(0) = 1 \qquad f(x) = f(-x) \qquad \alpha = 1/f(1).$$

Find an approximate value α for the universal constant $\alpha^* = 1/f^*(1)$, where $Rf^* = f^*$. Specifically, let $f(x) = 1 - \lambda x^2$, and determine λ such that

$$(Rf)(x) = f(x) + O(x^4)$$

whence the corresponding value of α. Compare the latter with α^*.

Solutions

Solution 1. We consider a real map, depending on a parameter. Its stable cycles are points on the real line. As the parameter changes, these points also change, and in the parameter-coordinate plane we obtain a graph, called a bifurcation tree.

The bottom branch of the tree corresponds to the fixed point, while the nodes correspond to period-doubling bifurcations, where a cycle loses stability, giving birth to a cycle of twice the period. The leaves correspond to the accumulation point of the period-doubling sequence.

Solution 2. We consider a renormalisation operator, acting by function composition combined with scaling about the origin. The Feigenbaum-Citanović equation is the fixed point equation of such operator, and its solutions are functions which are invariant under composition and scaling. One typically looks for analytic solutions satisfying certain normalisation conditions.

For instance, restriction to even analytic functions with a quadratic maximum at the origin, and whose value at zero is unity, yields the universal function of the accumulation point of period-doubling. The associated scaling constant (the reciprocal of the value of the fixed point at 1), is a universal constant.

Solution 3.

(a) The fixed points satisfy the equation $f(x) = x$, so there are two cases:

i) $0 \le x < 1/2$. We have $x + 2x^2 = x$, hence $x = 0$. We find $f'(x) = 1 + 4x$ so $f'(0) = 1$, and we must look at higher-order terms. Since $f''(0) = 4 > 0$, we have that for $x > 0$, $f(x) > x$, and the fixed point is unstable.

ii) $1/2 < x \le 1$. We have $x = 2 - 2x$, giving $x = 2/3$. We have $f'(x) = -2$ for all x in that range, so the fixed point is unstable.

(b) Let $(x_1,\ldots,x_n)$ be a n-cycle. We may assume $n>1$, since the fixed points have been dealt with already. No point in the n-cycle can be equal to $1/2$, since $f^2(1/2)=0$, and so $1/2$ is non-periodic (it is eventually fixed). In particular, f is differentiable at every point in the cycle. The multiplier of the cycle is given by the product

$$f'(x_1)\,f'(x_2)\cdots f'(x_n).$$

For all point x in the cycle we have $x\neq 0,1/2$. Thus the derivative of f is defined, and is greater than 1 in absolute value. It follows that any n-cycle is unstable.

(c) A 2-cycle must necessarily have one point x_1 in the interval $(0,1/2)$ and another point x_2 in the interval $(1/2,1)$. Let $x\in(0,1/2)$. Then $f(x)=x+2x^2$, and $f^2(x)=2-2\,(x+2x^2)$. The equation $f^2(x)=x$ gives

$$4x^2+3x-2=0 \qquad\Longrightarrow\qquad x_1=\frac{-3\pm\sqrt{41}}{8}$$

and we must choose the positive sign. Then

$$x_2=f(x_1)=\frac{19-\sqrt{41}}{16}.$$

The derivative of f at x_1 and x_2 is equal to $1-4x_1$ and -2, respectively. So the multiplier of the 2-cycle is

$$2\,(1-4x_1)=5-\sqrt{41}<-1.$$

Solution 4.

(a) We construct f^2 using the left branch $f(x)=x+2x^2$, since if x is small, $f(x)<1/2$. We find

$$\alpha\,f(f(\alpha))=x+4\,\frac{x^2}{\alpha}+8\,\frac{x^3}{\alpha^2}+8\,\frac{x^4}{\alpha^3}.$$

Choosing $\alpha=2$ gives

$$2\,f(f(x/2))=x+2x^2+O(x^3)=f(x)+O(x^3)$$

as desired.

(b) Letting $\hat{f}(x) = x + 2x^2 + cx^3$ we find

$$2\,\hat{f}(\hat{f}(x/2)) = x + 2x^2 + \left(2 + \frac{c}{2}\right)x^3 + O(x^4) = \hat{f}(x) + O(x^4).$$

The last equation gives

$$2 + \frac{c}{2} = c$$

or $c = 4$. Thus the function $\hat{f}(x) = x + 2x^2 + 4x^3$ satisfies Feigenbaum-Cvitanović equation near zero, up to order 4.

(c) We find

$$\alpha g(g(x/\alpha)) = \alpha g\left(\frac{x/\alpha}{1 - bx/\alpha}\right) = \alpha \frac{\dfrac{x/\alpha}{1 - bx/\alpha}}{1 - b\,\dfrac{x/\alpha}{1 - bx/\alpha}} = \frac{x}{1 - 2bx/\alpha}.$$

So choosing $\alpha = 2$ we obtain

$$2g(g(x/2)) = g(x).$$

Note that, for sufficiently small x, we have

$$g(x) = \frac{x}{1 - bx} = x(1 + bx + b^2x^2 + \cdots) = x + bx^2 + b^2x^3 + b^3x^4 + \cdots$$

Thus the functions f and $\hat{f}$ of the previous problem are two successive approximations to this solution for $b = 2$.

Solution 5. Let $x_0 = 0$, the critical point. Then $x_2 = f^2(0) = f(1) = 1 - \lambda < -1$, since $\lambda > 2$. We have $f(-1) = 1 - \lambda = -1 - c$, with $c = -2 + \lambda > 0$. Furthermore, $f'(x) = -2\lambda x > 1$ for all $x < -1$. Therefore, for all $x < -1$ we have $f(x) < g(x)$, where $g(x) = x - c$. An easy induction shows that $g^t(x) = x - ct$, and therefore, for $x < -1$, we have $f^t(x) < x - ct$.

Putting everything together, we have $f^{t+2}(0) = f^t(x_2) < 1 - \lambda - ct \to -\infty$, as $t \to \infty$.

Solution 6. We find that the functions $f(x) = \alpha e^x$, where α a real number, form a one-parameter family of fixed points of D.

The set $\{e^{-x}, -e^{-x}\}$ is a 2-cycle.

The set $\{\sin(x), \cos(x), -\sin(x), -\cos(x)\}$ is a 4-cycle.

Let $g(x) = f(x) + p(x)$, where $f(x)$ is periodic and $p(x)$ is a polynomial. Then $g(x)$ is eventually periodic. The degree of $p(x)$ gives the transient length.

Solution 7. We have

$$(Rf)(0) = \alpha\, f(0\cdot\alpha) = \alpha\, f(0) = 0.$$

Let ρ_f be the radius of convergence of the series

$$f(x) = \sum_{k=0}^{\infty} c_k x^k$$

or

$$\frac{1}{\rho} = \limsup_{k\to\infty} (c_k)^{1/k} < \infty.$$

Let now $\hat{f} = \alpha\, f(x/\alpha)$ where

$$\hat{f}(x) = \sum_{k=0}^{\infty} \hat{c}_k x^k.$$

We find

$$\hat{c}_k = \frac{c_k}{\alpha^{k-1}}$$

and therefore

$$\frac{1}{\hat{\rho}} = \limsup_{k\to\infty} (\hat{c}_k)^{1/k} = \limsup_{k\to\infty} \left(\frac{1}{\alpha^{k-1}}\right)^{1/k} c_k^{1/k} = \frac{1}{\rho} \limsup_{k\to\infty} \frac{1}{\alpha^{(k-1)/k}} = \frac{1}{\alpha\rho}.$$

Thus

$$\hat{\rho} = \alpha\rho.$$

Under repeated applications of R, the radius or convergence diverges to infinity.

Solution 8. Let $f(x) = 1 - \lambda x^2$. Then $\alpha = 1/f(1) = 1/(1-\lambda)$, and

$$(Rf)(x) = \frac{1}{\lambda - 1} f(1 - \lambda(1-\lambda)^2 x^2) = 1 + 2\lambda^2(1-\lambda)x^2 + O(x^4).$$

Requiring that $(Rf)(x) = f(x) + O(x^4)$ yields $2\lambda^2 - 2\lambda - 1 = 0$, or

$$\lambda = \frac{1+\sqrt{3}}{2} \qquad \alpha = -1 - \sqrt{3} = -2.732\ldots$$

to be compared with the exact value $\alpha^* = -2.50290\ldots$.

Chapter 5

Homeomorphisms and diffeomorphisms

Exercise 1. Consider the following maps $f : \mathbb{R} \to \mathbb{R}$ with $f(x)$ given by

$$\begin{array}{llllll} a) & -2x+7 & b) & -x^2 & c) & x^3 \\ d) & e^x & e) & -2x-\sin(x) & f) & x+\sin(x) \end{array}$$

Decide in each case if f is a homeomorphisms, a diffeomorphisms, or neither. If f is a hom(diff)eo, decide if it is order-preserving or reversing.

Exercise 2. We represent the circle $\mathbb{S}^1$ as the interval $[-1,1]$ with the end-points identified. Consider the following family of mappings

$$f_\lambda : \mathbb{S}^1 \to \mathbb{S}^1 \qquad f_\lambda(x) = \lambda x + (1-\lambda)x^3, \qquad \lambda \in \mathbb{R}.$$

Determine the ranges of values of λ for which f_λ is $i)$ a homeomorphism; $ii)$ a C^r-diffeomorphism (in which case you must determine r); $iii)$ neither.
[*Note that both domain and co-domain are circles, so the points* -1 *and* $+1$ *must be identified. In all, there are* <u>seven</u> *distinct* λ*-ranges to be considered.*]

Exercise 3. Construct orientation-preserving diffeomorphisms $f : \mathbb{R} \to \mathbb{R}$ with the properties specified below. (First draw the graph of a function with the desired properties. Then define it analytically, and prove that it satisfies such properties.)

(a) A single attractive fixed point at $x = -1$.

(b) A single fixed point at $x = 0$, which is neither attractive nor repelling.

(c) Three fixed points —two attractors and one repeller.

Exercise 4. Construct an orientation-reversing diffeomorphisms $f : \mathbb{R} \to \mathbb{R}$ with the following properties.

(a) A single unstable fixed point at $x = 1$.

(b) A single symmetric 2-cycle, that is, a 2-cycle of the form $\{x_0, -x_0\}$.

Exercise 5. We wish to construct an orientation-reversing diffeomorphism of a neighbourhood of the origin, *all* whose points are 2-cycles (we ignore the trivial case $f(x) = -x$). The strategy is to build $f(x)$ as a power series, in such a way that $f^2(x) = x + O(x^k)$, for increasing k.

(a) Let $f(x) = -x + x^2$. Verify that $f^2(x) = x + O(x^3)$.

(b) Let $f(x) = -x + x^2 + \alpha x^3$, where α is a real parameter. Determine α so that $f^2(x) = x + O(x^5)$.

(c) Proceed as above to determine the coefficient of terms of higher degree, up to the largest degree you can handle (you may need Maple). The larger the degree, the larger the mark.

Note: the above method yields only a *formal* solution, since the power series constructed in this way may have a zero radius of convergence.

Solutions

Solution 1.

a) Order-reversing diffeo.

b) Not invertible, so neither.

c) Order-preserving homeo, but not diffeo since f^{-1} is not differentiable at zero.

d) Not surjective, so neither.

e) Order-reversing diffeo.

f) Order-preserving homeo, because the derivative vanishes.

Solution 2. We compute

$$f'(x) = \lambda + 3\,(1-\lambda)\,x^2 \qquad\qquad f''(x) = 6\,(1-\lambda)\,x.$$

As a real function, the function f is analytic. We check its behaviour at the points $x = \pm 1$. We find $f(1) = 1$ and $f(-1) = -1$, and since 1 and -1 are identified, f is continuous on $\mathbb{S}^1$. Furthermore, $f'(1) = f'(-1) = 3 - 2\lambda$, whereas $f''(1) = -f''(-1) = 6(1-\lambda)$. Therefore f is of class C^1 but not C^2, for all values of λ, except for $\lambda = 1$ (see below).

Furthermore

$$f'(x) = 0 \qquad \Longleftrightarrow \qquad x^2 = \frac{\lambda}{3(\lambda-1)}.$$

So $f'(x)$ does not vanish at all, or does not vanish in the interval $[-1,1]$, respectively, in the ranges

$$0 < \lambda < 1 \qquad\qquad 1 < \lambda < \frac{3}{2}.$$

There are three critical values.

i) $\lambda = 0$. The function $f_0(x) = x^3$ is of class C^0.

ii) $\lambda = 1$. The function $f_1(x) = x$ is of class C^∞.

iii) $\lambda = 3/2$. The function $f_{3/2}(x) = (3x - x^3)/2$, is of class C^0.

SUMMARY:

λ	*class*	*comment*
$(-\infty, 0)$	$-$	f not invertible
0	C^0	f^{-1} not differentiable at $x = 0$
$(0, 1)$	C^1	f'' not continuous at $x = 1$
1	C^∞	$f(x) = x$
$(1, 3/2)$	C^1	f'' not continuous at $x = 1$
$3/2$	C^1	f^{-1} not differentiable at $x = 1$
$(3/2, \infty)$	$-$	f not invertible

Solution 3.

(a) We let $\Phi_1(x) = f(x) - x = \alpha\,(x+1)$, giving $f(x) = x(\alpha+1) + \alpha$. Choosing $\alpha > -1$ we ensure the orientation-preserving property, while for $-2 < \alpha < 0$ the point $x = -1$ is an attractor. Thus $-1 < \alpha < 0$.

(b) The function

$$g(x) = \frac{1}{x^2+1} \qquad g'(x) = -2\,\frac{x}{(x^2+1)^2} \qquad g''(x) = 2\,\frac{3x^2-1}{(x^2+1)^3}$$

has a single maximum at $x=0$, with $g(0)=1$. The second derivative $g''(x)$ vanishes at $x=\pm 1/\sqrt{3}$, so that

$$|g'(x)| \leq |g'(1/\sqrt{3})| = \frac{8}{9\sqrt{3}} < 1$$

Moreover $g''(0) < 0$. Therefore the function

$$f(x) = x - 1 + g(x)$$

has the following properties:

$i)\quad f'(x) > 0 \qquad ii)\quad f(0) = 0 \qquad iii)\quad f'(0) = 1 \qquad iv)\quad f''(0) < 0.$

Now, the function f is differentiable (because g is). $i)$ says that f is an orientation-preserving diffeomorphism. $ii)$ says that f has a fixed point at zero. $iii)$ and $iv)$ say that such a point is neither an attractor nor a repeller.

(c) Let α be a positive real number, and let

$$f(x) = \alpha x + \arctan(x) \qquad f'(x) = \alpha + \frac{1}{x^2+1} > 0.$$

Then f is an orientation-preserving diffeomorphism of the real line. The fixed point equation reads

$$\Phi_1(x) = f(x) - x = (\alpha-1)x + \arctan(x) = 0.$$

We have $\Phi_1(0) = 0$, so $x=0$ is a fixed point. Furthermore

$$\Phi_1(x) = \alpha x + O(x^2), \quad x \to 0; \qquad \Phi_1(x) = (\alpha-1)x + O(1), \quad x \to \infty.$$

Let $0 < \alpha < 1$. Then $\Phi_1(x)$ is positive for sufficiently small x and negative for sufficiently large x, and from the intermediate value theorem, there exists a point $x^* > 0$ for which $\Phi_1(x^*) = 0$. Such a point is unique, since $\Phi_1''(x) < 0$ for all positive x. The third fixed point $x = -x^*$ then results from symmetry: $\Phi_1(x) = -\Phi_1(-x)$. Since $f'(0) = \alpha + 1 > 0$, the origin is a repeller. Since $0 < f'(x^*) = f'(-x^*) < 1$, the other points are attractors.

Solution 4.

(a) We let $\Phi_1(x) = f(x) - x = \alpha(x+1)$, giving $f(x) = x(\alpha+1) + \alpha$. We need $\alpha < -1$ for reversing the orientation and $|\alpha+1| > 1$ for a repeller, so $\alpha < -2$.

(b) Consider the analytic function

$$f(x) = -x + \varepsilon g(x) \qquad\qquad g(x) = (x^2 - x_0^2)\,e^{-x^2}.$$

We have $f(\pm x_0) = \mp x_0$, so $\{x_0, -x_0\}$ is a 2-cycle. The function $g(x)$ is differentiable with *bounded* derivative $|g'(x)| < C$, say. Then $f'(x) > 0$ for all $|\varepsilon| < 1/C$. Since $g(x)$ does not vanish at any other point, $f(x)$ does not have any other symmetric 2-cycle.

(Generalisation. The function

$$f(x) = -x + \varepsilon g(x) \qquad\qquad g(x) = \prod_{i=1}^{n}(x^2 - x_i^2)\,e^{-x^2}$$

has precisely n symmetric 2-cycles. Furthermore, for sufficiently small $|\varepsilon|$, f is a diffeomorphism.)

Solution 5.

(a) We have

$$f^2(x) = x - 2x^3 + x^4 = x + O(x^3).$$

(b) Let $f(x) = -x + x^2 + \alpha x^3$. We find

$$\begin{aligned} f^2(x) &= x - x^2 - \alpha x^3 + (-x + x^2 + \alpha x^3)^2 + \alpha(-x + x^2 + \alpha x^3)^3 \\ &= x - 2(\alpha+1)x^3 + (\alpha+1)x^4 + \alpha(-1 + 3\alpha)x^5 + O(x^6). \end{aligned}$$

Choosing $\alpha = -1$ we kill both the cubic and quartic term, giving

$$f^2(x) = x + 4x^5 + O(x^6) = x + O(x^5).$$

(c) The function

$$\begin{aligned} f(x) = \sum_{k=1}^{20} c_k x^k &= -x + x^2 - x^3 + \frac{2}{3}x^4 - 1736x^6 + \frac{533}{540}x^8 - \frac{43981}{12960}x^{10} \\ &\quad + \frac{1111801}{68040}x^{12} - \frac{558379369}{5443200}x^{14} + \frac{527406923}{653184}x^{16} \\ &\quad - \frac{10155352946783}{1306368000}x^{18} + \frac{17448102987228961}{193995648000}x^{20}. \end{aligned}$$

has the property that

$$f^2(x) = x + O(x^{23}).$$

This expansion is almost certainly divergent, as demonstrated by the the regular growth of $(c_k)^{1/k}$ with k.

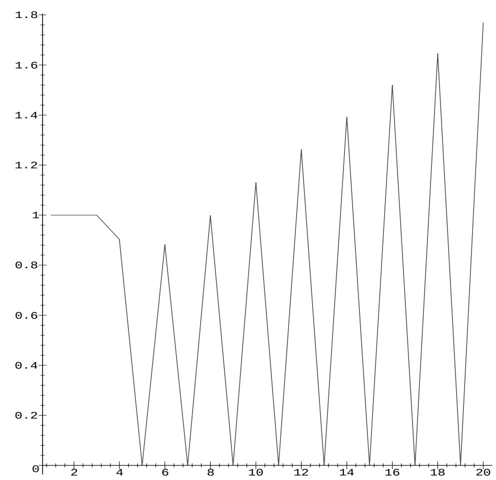

Figure 5.1: Growth of the coefficients c_k.

Chapter 6

Doubling map

Exercise 1. Order the integers from 20 to 30 inclusive, using Sharkowsky's ordering. Then explain what it means.

Exercise 2. Consider the doubling map.

(a) By looking at periodic binary digits, show that there are three orbits of minimal period 4. How many orbits are there of minimal period 6?
[*To answer, you do not need to compute all 6-strings!*]

(b) Determine all points of the 3-cycle with initial condition

$$x_0 = .001\,001\,001\,001\ldots = .\overline{001}$$

as rational numbers.
[*The number x_0 is the sum of a geometric series.*]

(c) Do the same for the 6 cycle

$$x_0 = .001101\,001101\,001101\ldots = .\overline{001101}.$$

(d) Divide the unit interval in four equal sub-intervals, hence determine how the points of the periodic orbit with initial condition $1/51$ are distributed among those intervals. Do the same with the orbit with initial condition $1/13$.

(e) Divide the unit interval in 16 equal sub-intervals, hence determine the binary digits of a 16-cycle that has has one point in each sub-interval.

(f) Let x be a point in an n-cycle of the doubling map

$$x = \sum_{k=1}^{\infty} b_k 2^{-k} \qquad\qquad x_k = x_{k+n}.$$

Prove that

$$x = \frac{B}{2^n - 1} \qquad \text{where} \qquad B = B(b_1, b_2, \ldots, b_n) = \sum_{k=1}^{n} b_k\, 2^{n-k}.$$

[*Write* $k = sn + r$, *with* $1 \le r \le n$, *then split the summation into two parts.*]

Exercise 3. Consider the mapping $f(x) = 1 - 2\sqrt{|x|}$ on the interval $[-1, 1]$.

(a) Draw the graph of f hence determine f'. Is this map chaotic?
[*This is a tricky question. Consider the orbit with initial condition* $x = 0$ *separately. You would need a positive lower bound for the Lyapounov exponent, but that's hard to obtain.*]

(b) Theory shows that the average position of the points in a 'typical' non-periodic orbit is equal to $-1/3$, namely that

$$\bar{x} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} x_t = -\frac{1}{3}.$$

Compute numerically the average position of the points of an orbit, using an initial condition x_0 of your choice and the largest value of N your computer can handle[1]. Verify that this result is in agreement with the theory (Do not expect great accuracy: convergence is slow.)

Solutions

Solution 1.

$$24 \lhd 28 \lhd 20 \lhd 30 \lhd 26 \lhd 22 \lhd 29 \lhd 27 \lhd 25 \lhd 23 \lhd 21$$

If a continuous map $f : [a, b] \to \mathbb{R}$ has an orbit of one of the above periods, then it also has orbits of all the periods that are "smaller" according to this ordering. In particular, if the period 21 is present, so are all the other periods in the list.

[1] If $N < 1000$, then it is time to junk your computer.

Solution 2.

(a)

$$0011, 0001, 0111.$$

There are 2^6 binary strings of length 6. Of those, 2^3 have minimal period 3 or 1, which must be subtracted. Furthermore 2^2 strings have minimal period 2 or 1, which must also be subtracted, except that in doing so we subtract the 1-strings twice. Thus the total number of strings of minimal period 6 is

$$2^6 - 2^3 - 2^2 + 2^1 = 64 - 8 - 4 + 2 = 54.$$

The total number of *orbits* is then given by $54/6 = 9$.

(b) We have

$$x_0 = .\overline{001} = \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots = \sum_{k=1}^{\infty} \frac{1}{8} = \frac{1}{7}.$$

Then

$$x_1 \equiv 2x_0 \,(\text{mod } 1) = \frac{2}{7} \qquad\qquad x_2 \equiv 2x_1 \,(\text{mod } 1) = \frac{4}{7}.$$

(c) We have

$$\begin{aligned} x_0 &= .\overline{001101} = .\overline{000001} + .\overline{000100} + .\overline{001000} \\ &= .\overline{000001} \times (1+4+8) \\ &= 13 \sum_{k=1}^{\infty} \left(\frac{1}{2^6}\right)^k = \frac{13}{63}. \end{aligned}$$

Iterating the map, we find

$$x_0 = \frac{13}{63}, \quad x_1 = \frac{26}{63}, \quad x_2 = \frac{52}{63}, \quad x_3 = \frac{41}{63}, \quad x_4 = \frac{19}{63}, \quad x_5 = \frac{38}{63}.$$

(d) Let $x_0 = 1/51$. The numerators of the points in the orbits are

$$1, 2, 4, 8, 16, 32, 13, 26, 1, \ldots$$

So the orbit has period 8. We partition the unit interval into four sub-intervals

$$I_k = \big[(k-1)/4, k/4\big) \quad k = 1, \ldots, 4 \qquad\qquad \bigcap_{k=1}^{4} I_k = [0, 1).$$

We have (referring to numerators only)

$$\{1,2,4,8\} \in I_1 \qquad \{13,16\} \in I_2 \qquad \{26,32\} \in I_3.$$

Likewise, if $x_0 = 1/13$ we find

$$1,2,4,8,3,6,12,11,9,5,10,7$$

so the period is 12. This time there are 3 points in each interval, so the orbit is uniform.

(e)

$$x_0 = \overline{0000111100101101} = 259/4369.$$

The above string contains all 4-substrings. (There is more than one orbit with this feature.)

(f) Let the binary representation of x have n periodic digits

$$x = .\overline{b_1 \cdots b_n}$$

Then

$$x = \sum_{k=1}^{\infty} b_k 2^{-k} \qquad b_k = b_{k+n},\ \forall k > 0.$$

By writing $k = sn + r$, with $1 \le r \le n$, we obtain $b_k = b_r$, from the above equation, and we can write

$$\begin{aligned} x &= \sum_{s=0}^{\infty} \left[\sum_{r=1}^{n} b_r 2^{-r} \right] 2^{-ns} \\ &= \sum_{r=1}^{n} b_r 2^{-r} \cdot \sum_{s=0}^{\infty} \left(2^{-n}\right)^s \\ &= \sum_{r=1}^{n} b_r 2^{-r} \frac{1}{1-2^{-n}} \\ &= 2^n \sum_{r=1}^{n} b_r 2^{-r} \frac{1}{2^n - 1} \\ &= \sum_{r=1}^{n} b_r 2^{n-r} \frac{1}{2^n - 1}. \end{aligned}$$

Solution 3.

(a) Let $y = f(x) = 1 - 2\sqrt{|x|}$. We find

$$|x| = \left(\frac{1-y}{2}\right)^2 \qquad\qquad x = \pm\left(\frac{1-y}{2}\right)^2.$$

The map is not differentiable at $x = 0$. We find

$$f'(x) = \begin{cases} \dfrac{1}{\sqrt{|x|}} & x < 0 \\ -\dfrac{1}{\sqrt{|x|}} & x > 0. \end{cases}$$

The orbit of 0 of pre-periodic: $0 \mapsto 1 \mapsto -1 \mapsto -1$. Apart from ± 1, the map is differentiable with $|f'(x)| > 1$, so the map appears to be chaotic. However, to prove it in a straightforward way, we would need a positive lower bound for the Lyapounov exponent. However, the quantity $\log|f'(x)|$ is not bounded away from zero.

(b) We compute

$$\Lambda_N(x_0) = \frac{1}{N}\sum_{t=0}^{N-1} x_t.$$

```
# ---- the function f
f:=x->1-2*sqrt(abs(x)):
# ---- parameters
N:=1000:
nprint:=5:
# ---- initial condition
x:=1.0/3.0:
xbar:=x:
# ---- main loop
for s to nprint do
   to N do
     x:=f(x):
     xbar:=xbar+x
    od:
    print(xbar/(s*N))
```

`od:`

Selected output for $x_0 = 1/3$:

N	Λ_N
1000	-0.3450708675
2000	-0.3573080697
3000	-0.3362268846
4000	-0.3128255468
5000	-0.3164218084

Convergence is slow.

Chapter 7

Conjugacy

Exercise 1. What is chaos? Explain it briefly, without using symbols.

Exercise 2. Write $f \sim g$ if f is topologically conjugate to g. Prove that $\sim$ is an equivalence relation, i.e., $\sim$ is reflexive, symmetrical and transitive.

Exercise 3. Let f and g be conjugate via a diffeomorphism h. Let f have a fixed point at x^*.

(a) Prove that the multiplier of g at $h(x^*)$ is the same as that of f at x^*.

(b) Formulate and prove the analogous statement for an n-cycle.

Exercise 4. Consider the following mappings

$$f(x) = ax \qquad g(y) = by \qquad (a \neq b).$$

(a) Determine conditions on a and b for which f and g are conjugate via $y = h(x) = x^n$, where n is a positive integer.

(b) Use problem 2 (a) to infer that no diffeomorphism exists which conjugates f and g.

Exercise 5. Let $f(x) = 2x(1-x)$, and let a, b be distinct complex numbers. Construct a map $g(y)$ which is conjugate to f, and which has fixed points at $y = a$ and $y = b$ (with the former superstable).

Exercise 6. Consider the complex mapping

$$f(z) = \frac{z^2 + n}{2z} \qquad n \in \mathbb{Z}.$$

(a) Show that f is the Newton's iteration for finding the solutions of the equation $z^2 - n = 0$.

(b) Show that f is conjugate to

$$g(w) = w^2$$

via

$$w = \phi(z) = \frac{z + \sqrt{n}}{z - \sqrt{n}}.$$

(c) Using the above conjugacy, what can you say about the periodic orbits of f, and their stability?

Solutions

Solution 1. Chaos is sensitive dependence on initial conditions, a mechanism that causes the orbits of a dynamical system which are initially close to each other to separate exponentially. Chaotic systems features a dense set of unstable periodic orbits.

Formally, a system is defined to be chaotic if its Lyapounov exponent is positive. This is the average of the logarithm of the multiplier along an orbit, which turns out to be the same for almost all orbits.

A most prominent feature of chaotic behaviour is the emergence of probabilistic laws, which exist alongside the extremely complex orbital motions.

Solution 2. Notation:

$$f \overset{h}{\sim} g \qquad \Longleftrightarrow \qquad h \circ f = g \circ h$$

Reflexivity: $f \sim^{id} f$.

Simmetry: $f \sim^{h} g$ implies $h^{-1} \circ g = f \circ h^{-1}$, that is, $g \sim^{h^{-1}} f$. Now h^{-1} is a homeo, by definition.

Transitivity: if $f \sim^{h} g$ and $g \sim^{k} l$, then

$$f = h^{-1} \circ g \circ h = h^{-1} \circ k^{-1} \circ l \circ k \circ h = (k \circ h)^{-1} \circ l \circ (k \circ h).$$

Now, the composition of continuous bijections is a continuous bijection, so $(k \circ h)$ is a homeomorphism, and $f \sim^{k\circ h} l$.

Solution 3.

(a) We have $g = h \circ f \circ h^{-1}$. Hence, letting $y = h(x)$, we have

$$g'(y) = (h^{-1})'(y)\, f'(x)\, h'(f(x))$$

If x is a fixed point, $f(x) = x$, and therefore

$$g'(y) = f'(x)[(h^{-1})'(y)\, h'(x)]$$

but the product in square brackets is unity, because

$$1 = (h^{-1} \circ h)'(x) = h'(x)\,(h^{-1})'(y).$$

(b) *Let h be a smooth conjugacy between f and g, and let $\{x_k^*\}_{k=1}^n$ be an n cycle for f. Then $\{h(x_k^*)\}_{k=1}^n$ is an n-cycle for g, with the same multiplier.*

For an n-cycle, first note that if $f \sim g$, then $f^n \sim g^n$. (with the same conjugacy function —see notes). Furthermore we know (see notes) that $\{h(x_k^*)\}_{k=1}^n$ is an n-cycle for g. Then proceed as above with f^n and g^n in place of f and g, respectively.

Solution 4.

(a) We have

$$h(f(x)) = (ax)^n = g(h(x)) = bx^n.$$

So $b = a^n$, with n odd. This is a homeomorphism, but not a diffeomorphism.

(b) It suffices to note that f and g both have a unique fixed point, but with different multiplier.

Solution 5. The mapping f as an unstable fixed point at $x = 0$ and a superstable fixed point at $x = 1/2$.

Let $h(x) = mx + q$. Requiring $h(0) = b$ and $h(1/2) = a$ yields

$$h(x) = 2(a-b) + b \qquad h^{-1}(y) = \frac{y-b}{2\,(a-b)}.$$

The equation $g(y) = h(f(h^{-1}(y)))$ gives

$$g(y) = \frac{y^2 - 2ay + ab}{b-a}$$

which is the desired mapping.

Solution 6.

(a) The Newton's map for finding the roots of the equation $h(z) = z^2 - n = 0$ is

$$f(z) = z - \frac{h(z)}{h'(z)} = z - \frac{z^2-n}{2z} = \frac{z^2+n}{2z}.$$

(b) Two mappings $f : X \to X$ and $g : Y \to Y$ are (topologically) *conjugate* if there exists a homeomorphism (bi-continuous map) $\phi : X \to Y$ such that $\phi \circ f = g \circ \phi$.

The map ϕ is a homeomorphism of the Riemann sphere. We prove that $\phi \circ f = g \circ \phi$.

$$(\phi \circ f)(z) = \frac{z^2+n+2z\sqrt{n}}{z^2+n-2z\sqrt{n}} = \left(\frac{z+\sqrt{n}}{z-\sqrt{n}}\right)^2 = (\phi(z))^2 = (g \circ \phi)(z).$$

(c) The restriction of g to the unit circle is the doubling map, and hence g has infinitely many periodic orbits, of every period. All these cycles are unstable, and they are dense on the unit circle, which is the Julia set of g. In addition, g has two superstable fixed points, at 0 and ∞,

It can be shown that a smooth conjugacy preserves periodic orbits and their multipliers.

Because the conjugacy ϕ is smooth, the map f has the same orbit structure as g. (One verifies that $J(f)$ is $\mathbb{R}$ if $n < 0$, and $\sqrt{-1}\mathbb{R}$ if $n > 0$.)

Chapter 8

Two-dimensional mappings

Exercise 1. Consider the following two-dimensional map:

$$(x,y) \mapsto (f(x,y), g(x,y)).$$

Write down the equations whose solutions are the points of period dividing 3.

Exercise 2. Consider the following map of $\mathbb{R}^2$:

$$\Phi(x_t, y_t) = (x_{t+1}, y_{t+1})$$

where

$$\begin{aligned} x_{t+1} &= -y_t + g(x_t) \\ y_{t+1} &= x_t - g(x_{t+1}) \end{aligned}$$

and g is a real function.
[*Check the subscripts carefully.*]

(a) Find a function g for which Φ has no fixed points.
[*This means that the fixed-point equation has no solutions.*]

(b) Let $g(x) = \lambda + x^2$. Determine the range of values of λ for which Φ has two distinct fixed points.

Exercise 3. Consider the two-dimensional mapping

$$\begin{aligned} x_{t+1} &= x_t + \frac{2\pi\lambda}{y_{t+1}} = x_t + \frac{2\pi\lambda}{y_t + \sin x_t}; \\ y_{t+1} &= y_t + \sin x_t. \end{aligned}$$

where x, y, λ are real numbers, $\lambda > 0$, $y_t \neq 0$, and x_t is periodic with period 2π. (Thus the phase space is a cylinder, with the circle $y = 0$ removed.)

(a) Show that the mapping has two infinite families of fixed points.

(b) Show that, for fixed λ, the number of (marginally) stable fixed points is at most finite. Determine the range of values of λ for which all fixed points are unstable.

Exercise 4. Consider the linear map

$$f : \mathbb{C} \to \mathbb{C} \qquad\qquad f(z) = \lambda z.$$

Prove that all orbits of f are periodic if and only if λ has unit modulus, and its argument is a rational multiple of 2π.

Solutions

Solution 1.

$$\begin{aligned} x &= f(f(f(x,y),g(x,y)),g(f(x,y),g(x,y))) \\ y &= g(f(f(x,y),g(x,y)),g(f(x,y),g(x,y))). \end{aligned}$$

Solution 2.

(a) Letting $x_t = x_{t+1} = x$ and $y_t = y_{t+1} = y$ and adding the two equations, we find

$$x + y = -y + x$$

giving $y = 0$, hence $x = g(x)$. So the fixed points of Φ take the form $(x^*, 0)$, where x^* is a fixed point of g. Therefore if g has no fixed point, neither has Φ. We choose $g(x) = x + 1$, say.

(b) The equation $g(x) = x$ gives $\lambda + x^2 = x$, and hence

$$x = \frac{1 \pm \sqrt{1-4\lambda}}{2}.$$

For two distinct real solutions, the discriminant must be positive, whence $\lambda < 1/4$.

Solution 3.

(a) The Jacobian matrix is given by

$$J(x,y) = \begin{pmatrix} 1 - \dfrac{2\pi\lambda\cos(x)}{(y+\sin x)^2} & -\dfrac{2\pi\lambda}{(y+\sin x)^2} \\ \cos(x) & 1 \end{pmatrix}.$$

One sees that $\mathrm{Det}(J(x,y)) = 1$ (the mapping is area-preserving).

The fixed points are:

$$\frac{2\pi\lambda}{y} = 2\pi k \qquad x \equiv 0, \pi (\mathrm{mod}\ 2\pi) \qquad k \in \mathbf{Z} \quad k \neq 0.$$

They form two families:

$$(0, \lambda/k) \qquad (\pi, \lambda/k) \qquad k \neq 0.$$

(b) We compute

$$J(\pi, \lambda/k) = \begin{pmatrix} 1 + \frac{2\pi k^2}{\lambda} & + \frac{2\pi k^2}{\lambda} \\ 1 & 1 \end{pmatrix}$$

$$\mathrm{Tr}(J(\pi, \lambda/k)) = 2 + \frac{2\pi k^2}{\lambda} > 2$$

so the points $(\pi, \lambda/k)$ are all unstable.

$$J(0, \lambda/k) = \begin{pmatrix} 1 - \frac{2\pi k^2}{\lambda} & - \frac{2\pi k^2}{\lambda} \\ 1 & 1 \end{pmatrix}$$

We have

$$\mathrm{Tr}(J(0, \lambda/k)) = 2 - \frac{2\pi k^2}{\lambda} < 2. \qquad (1)$$

For marginal stability, we require $\mathrm{Tr}(J(0,\lambda/k)) > -2$, that is,

$$\lambda > \frac{\pi k^2}{2}$$

which can be satisfied for at most finitely many values of k. The inequality (1) implies

$$|y| > \sqrt{\frac{\pi\lambda}{2}}.$$

which shows that the (marginally) stable fixed points are located sufficiently far away from the origin.

From (1) it also follows that for $\lambda < \pi/2$ there are no stable fixed points at all.

Solution 4. We have $z_n = \lambda^t z_0$, so if $z_0 = z_n$, then $|z_0| = |z_n| = |\lambda||z_0|$. If $z_0 \neq 0$, this gives $|\lambda| = 1$, as desired.

Let now

$$\lambda = e^{2\pi\alpha}.$$

For periodicity when $z \neq 0$, we need $\lambda^n = 1$. This gives $2\pi\alpha n = 2k\pi$, for some $k \in \mathbb{Z}$. But then $\alpha = k/n$, a rational number.

Conversely, if $|\lambda| = 1$ and the argument $2\pi\alpha$ of λ is such that $\alpha = k/n$, with k and n coprime, then n is the smallest natural number such that $\lambda^n = 1$, and all non-zero points are periodic with the same period n.

Chapter 9

Fractals

Exercise 1. Prove that the box dimension of the set

$$\{1/n^{\alpha} : n \in \mathbf{N}\} \qquad \alpha > 0$$

is equal to $1/(\alpha+1)$.

Exercise 2.

(a) Explain why in the definition of Hausdorff distance between two subsets of the plane, these sets are required to be closed and bounded.

(b) Compute the Hausdorff distance between the unit disc and a filled-in square inscribed in it (that is, the vertices of the square belong to the unit circle).

Exercise 3. Let $h(A,B)$ be the Hausdorff distance between two compact subsets of $\mathbb{R}$.

(a) Let C be the Cantor ternary set, let I be the closed unit interval, and let A be a set constituted by the point 0 and the rationals 3^{-k}, for $k = 0,1,\ldots$, that is

$$A = \left\{0,\, 1,\, \frac{1}{3},\, \frac{1}{3^2},\, \ldots\right\}.$$

Compute $h(I,C)$, $h(I,A)$ and $h(A,C)$.

(b) Determine iterated function systems whose attractors are I, C and A, respectively.

Exercise 4*. Let $\mathscr{H}(\mathbb{C})$ be the set of all closed and bounded subset of $\mathbb{C}$, with the Hausdorff distance. Let $\mathbb{S}$ be the unit circle in $\mathbb{C}$. Characterise the sets in $\mathscr{H}(\mathbb{C})$ which belong to the circle of unit radius and centre $\mathbb{C}$.

Exercise 5. Let A be the set of figure 9.

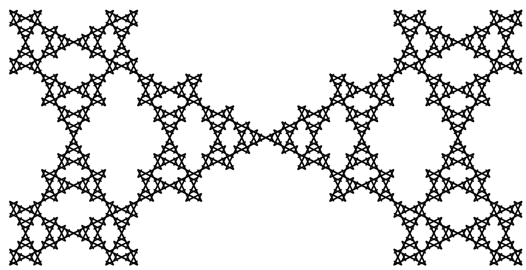

Figure 9.1: A fractal.

(a) After choosing coordinates, construct the iterated function system Φ whose fixed point is A.

(b) Compute the box dimension of A, hence show that A is a fractal.

(c) Write a Maple code to generate and plot this fractal.

Solutions

Solution 1. We choose boxes of size ε_n, where

$$\varepsilon_n = \frac{1}{n^\alpha} - \frac{1}{(n+1)^\alpha} \qquad \lim_{n\to\infty} \varepsilon_n = 0.$$

Using the binomial theorem, we find

$$\begin{aligned}\varepsilon_n &= \frac{(n+1)^\alpha - n^\alpha}{n^\alpha (n+1)^\alpha} = \frac{n^\alpha + \alpha n^{\alpha-1} + O(n^{\alpha-2}) - n^\alpha}{n^\alpha (n+1)^\alpha} \\ &= \frac{\alpha + O(n^{-1})}{n(n+1)^\alpha} \sim \frac{\alpha}{n^{\alpha+1}}\end{aligned}$$

where, if f and g are real functions, we write $f(n) \sim g(n)$ to mean

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

To cover the point $1, 1/2^\alpha, \ldots, 1/n^\alpha$, we need n boxes of size ε_n. The remaining points lie in the interval $[0, 1/(n+1)^\alpha]$. To cover such an interval, we need m boxes, where

$$m = \left\lceil \frac{1}{(n+1)^\alpha \varepsilon_n} \right\rceil \sim \frac{n}{\alpha}$$

and $\lceil \cdot \rceil$ is the ceiling function. Thus

$$N(\varepsilon_n) = n + m \sim n(1 + 1/\alpha).$$

We compute the box dimension

$$D = \lim_{n\to\infty} \frac{\log(N(\varepsilon_n))}{-\log(\varepsilon_n)} = \lim_{n\to\infty} \frac{\log(1+1/\alpha) + \log(n)}{-\log(\alpha) + (\alpha+1)\log(n)} = \frac{1}{\alpha+1}.$$

Thus, by adjusting α, the box dimension of our set can be any real number between 0 and 1.

Solution 2.

(a) Boundedness is required to avoid infinite distances. (Why? Give an example.) Closeness ensures that two sets are the same iff the distance between them is zero. (Why? Explain in detail.)

(b) If A is the disc and B the square, then $B \subset A$, and hence $h_{BA} = 0$. An elementary geometric consideration shows that

$$h_{AB} = h(A, B) = 1 - \frac{1}{\sqrt{2}}.$$

Solution 3.

(a) One finds, in a straightforward manner

$$h(I,C) = \frac{1}{6} \qquad h(I,A) = \frac{1}{3} \qquad h(A,C) = \frac{1}{3}.$$

(b) We look for contraction mappings covering the sets I, C, and A with smaller copies of themselves. One finds

$$\begin{array}{lll} I: & w_1(x) = x/2 & w_2(x) = x/2+1/2 \\ C: & w_1(x) = x/3 & w_2(x) = x/3+2/3 \\ A: & w_1(x) = x/3 & w_2(x) = 1. \end{array}$$

Note that the IFS for the set A requires condensation.

Hint 4. Any set lying inside $\mathbb{S}$ and containing the origin is at unit distance from it. Next find sets not contained in $\mathbb{S}$.

Solution 5.

(a) The set A is the union of three small copies of itself, two of which are rotated clockwise (or anti-clockwise) by $\pi/2$. It follows that the IFS for A comprises three mappings. Placing the origin in the barycentre of the A, and assuming that A has height 1, we find, using complex notation ($i = \sqrt{-1}$)

$$\begin{aligned} \Phi_1(z) &= \frac{z}{2} \\ \Phi_2(z) &= i\frac{z}{2} + \frac{3}{4} \\ \Phi_3(z) &= i\frac{z}{2} - \frac{3}{4}. \end{aligned}$$

(Alternatively, one could use affine maps of $\mathbb{R}^2$.)

(b) Because Φ consists of three maps, with the same contractivity factor $1/2$, it follows that the box dimension of A is $\log 3/\log 2$. Since 3 is not a power of 2, this number is not an integer. Hence, by definition, A is a fractal.

(c)
```
# ---- utilities
PlotStuff:=style=point,symbol=point,scaling=constrained,axes=none:
z2ReIm:=z->[Re(z),Im(z)]:
# ---- contraction mappings and IFS
half,threq:=evalf(1/2),evalf(2/3):
f1,f2,f3:=z->z*half,z->I*z*half+threq,z->I*z*half-threq:
Phi:=S->map(f1,S) union map(f2,S) union map(f3,S):
# ---- iterations and plot
{0}:to 8 do Phi(%) od:
plot(map(z2ReIm,%),PlotStuff);
```