Orbits of the doubling map

We consider again the doubling map

$$x_{t+1} \equiv 2x_t \pmod 1.$$

Let us expand x_t in base 2

$$x_t = \sum_{k=1}^{\infty} b_k \frac{1}{2^k} \qquad b_k \in \{0,1\}$$

or, more concisely,

$$x_t = .b_1 b_2 b_3 \ldots \ldots$$

Multiplication by 2 is the shift of the radix point

$$2x_t = b_1 . b_2 b_3 \ldots$$

while taking the remainder modulo 1 amounts to subtracting the integer part, that is dropping the digits to the left of the radix point

$$x_{t+1} = 2x_t \pmod 1 = .b_2 b_3 b_4 \ldots$$

and so the doubling map is represented by the (left) shift of binary digits

$$x_t = 0.b_1 b_2 b_3 b_4 \ldots$$

$$x_{t+1} = \underleftarrow{0.b_2 b_3 b_4 b_5 \ldots} = \sigma(x_t).$$

Thm An orbit of the doubling map is periodic if and only if the binary digits of its initial condition are periodic.

Pf If $x_0 = .\underbrace{b_1 b_2 \qquad b_T}\underbrace{b_1 b_2 \cdots b_T}\underbrace{b_1 \qquad}$ is periodic, then, after shifting T times

$$x_T = \sigma^T(x_0) = .\underbrace{b_1 b_2 \cdots\cdots b_T}\underbrace{b_1 \qquad} = x_0$$

So $x_T = x_0$, and the orbit is periodic. Conversely if $x_T = x_0$, then

$$x_T \equiv 2x_{T-1} \pmod 1 \equiv 2^2 x_{T-2} \pmod 1 \equiv \cdots \equiv 2^T x_0 \pmod 1$$

gives

$$x_0 \equiv 2^T x_0 \pmod 1 \Rightarrow x_0(2^T - 1) = m \in \mathbb{Z}$$

or

$$x_0 = \frac{m}{2^T - 1} \in \mathbb{Q}$$

where the denominator is odd. Now a rational number with odd denominator has periodic binary digits.(*) $\square$

Corollary Periodic orbits exist, of any given period.

(*) A special case of a more general result

i) $x \in \mathbb{Q} \iff$ base-b digits are eventually periodic

ii) $x \in \mathbb{Q} \cap [0,1)$ & denom of x coprime to b $\iff$ base-b digits are periodic.

<u>Lemma</u> Let $b_1^*, \dots, b_T^*$ be arbitrary binary digits. Then the set

$$I = \{x \in [0,1] \mid x = 0.b_1 b_2 \dots \ ; \ b_i = b_i^* \quad i = 1, \dots, T\}$$

is a closed interval of length 2^{-T}.

<u>Proof.</u> Let $\alpha = \sum_{k=1}^{T} b_k^* \frac{1}{2^k}$. Then $x \in I$ iff

$x = \alpha + \beta$ where $\beta = \frac{1}{2^T} \sum_{k=1}^{\infty} b_{T+k} \frac{1}{2^k}$.

The sum is an arbitrary real number between 0 (all $b_{T+k} = 0$) and 1 (all $b_{T+k} = 1$), and so β is an arbitrary number between 0 and $\frac{1}{2^T}$, inclusive.

Thus $$I = [\alpha, \alpha + 2^{-T}]$$

$\square$

Take an arbitrary open interval $\Delta \subset [0,1]$. Since the rational numbers of the form $\alpha = \sum_{k=1}^{T} b_k^* 2^{-k}$ are dense we may choose the b_k^* & T in such a way that $[\alpha, \alpha + 2^{-T}] \subset \Delta$.

Any initial condition of the form

$$x_0 = .b_1 b_2 \dots b_n \underbrace{b_1^* b_2^* \cdots b_T^*}_{} \dots$$

generates an orbit such that

$$x_n = .b_1^* b_2^* \cdots b_T^* \dots \in \Delta .$$

that is the orbit throug x_0 visits Δ. $\square$

This observation has far-reaching consequences. First of all, the periodic orbits of the doubling map are <u>dense</u> in $[0,1]$. Indeed if $x = .b_1 b_2 \ldots$ is an arbitrary point in $[0,1]$, then

$$x_0 = .\underbrace{b_1 b_2 \ldots b_T}\ \underbrace{b_1 b_2 \ldots b_T}$$

is a periodic point such that

$$|x_0 - x| \leq 2^{-T}$$

and T can be made as large as we please.

To see the broader picture, we introduce the following

<u>Def</u> Let $\alpha \in [0,1]$ be represented in base r, and let n_k be the number of times the digit $k \in \{0,1,\ldots,r-1\}$ occurs in the first n digits of α. Then α is said to be <u>normal</u> (in base r) if

$$\lim_{n\to\infty} \frac{n_k}{n} = \frac{1}{r}.$$

So α normal = every digit is equally represented. We have

<u>Theorem</u> (Borel 1909) Almost all numbers are normal in any base.

Note "Almost all" is meant with respect to the Lebesgue measure (see below).

Note that a string of digits in one base corresponds to just one digit in some other base. Thus

$$1001_2 = 9_{10}$$

thus almost all numbers contain all possible finite strings of digits infinitely often, and with the right frequency. So Borel's Thm has the following

<u>Corollary</u> Almost all orbits of the doubling map are dense in the unit interval, and spend equal time in equal subsets of the unit interval.

We shall return to the above result below.