2. Period-doubling bifurcations & transition to chaos

The logistic map:

$$x_{t+1} = 1 - \lambda x_t^2 = f_\lambda(x_t) \qquad (1)$$

$\lambda \in [0, 2]$; $x_0 \in [-1, 1] =: I$,

$$f_\lambda(I) \subseteq I \quad \text{for } \lambda \in [0, 2].$$

Fixed point: $x^* = f(x^*) \Rightarrow x^* = 1 - \lambda x^{*2}$

$$\lambda x^{*2} + x^* - 1 = 0 \qquad x^* = \frac{-1 \mp \sqrt{1+4\lambda}}{2\lambda}.$$

The fixed pt $x^* = \frac{-1+\sqrt{1+4\lambda}}{2\lambda} \in I$ for all $\lambda \in [0, 2]$. Indeed:

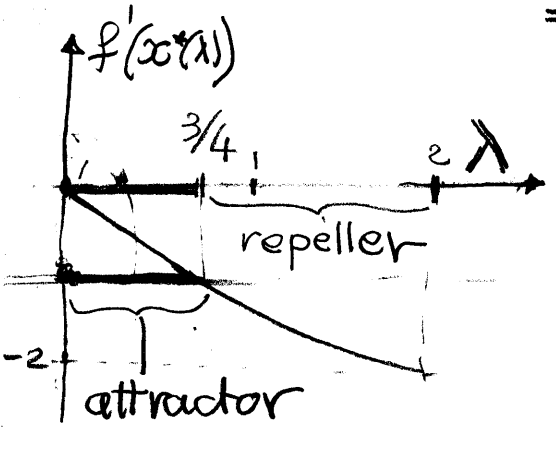
- $x^*(2) = \frac{1}{2}$
- $\lim_{\lambda \to 0} x^*(\lambda) = (\text{Hopital}) \lim_{\lambda \to 0} \frac{\frac{4}{2\sqrt{1+4\lambda}}}{2} = 1.$

In between, $x^*(\lambda)$ is monotonically decreasing.

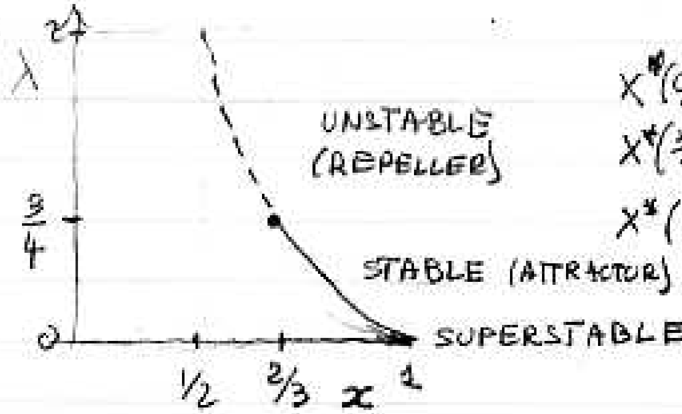
(The fixed pt $x^* = \frac{-1-\sqrt{1+4\lambda}}{2\lambda} \notin I$ in that parameter range.)

Stability

$$f'(x) = -2\lambda x, \text{ hence } f'(x^*) = -\cancel{2\lambda} \cdot \frac{-1+\sqrt{1+4\lambda}}{\cancel{2\lambda}}$$
$$= 1 - \sqrt{1+4\lambda}.$$



$$x^*(\lambda) = \begin{cases} \text{attr.} & 0 \le \lambda < 3/4 \\ \text{repell.} & \frac{3}{4} \le \lambda < 2. \end{cases}$$



$x^*(0)=1$
$x^*(3/4)=2/3$
$x^*(2)=1/2$

What happens at $\lambda = 3/4$? Must look at 2-cycles.

$$f^2(x)=x \quad \text{or} \quad P_2(x) = f^2(x)-x = f(f(x))-x = 0.$$

$$P_2(x) = 1-\lambda(1-\lambda x^2)^2 - x = -\lambda^3x^4 + 2\lambda^2x^2 - x - \lambda + 1$$

$$= \mp \underbrace{(x-x_0^*)(x-x_\pm^*)}_{\text{fixed points}}\underbrace{(x-x_+^*)(x-x_\mp^*)}_{\text{2-cycles}}$$

$$= \mp P_1(x)\cdot\frac{P_2(x)}{P_1(x)}$$

where $-P_1(x) = \lambda x^2 + x - 1$.

Compute $P_2(x)/P_1(x)$ by long division

$$\begin{array}{l|l}
\lambda^3x^4 \qquad -2\lambda^2x^2 + x + \lambda - 1 & \lambda x^2 + x - 1 \\
\underline{\lambda^3x^4 + \lambda^2x^3 - \lambda^2x^2} & \lambda^2x^2 - \lambda x + (1-\lambda) \\
\quad -\lambda^2x^3 - \lambda^2x^2 & \\
\quad \underline{-\lambda^2x^3 - \lambda x^2 + \lambda x} & \\
\qquad x^2(\lambda-\lambda^2) + x(1-\lambda) + (\lambda-1) & \\
\qquad \underline{x^2(\lambda-\lambda^2) + x(1-\lambda) + (\lambda-1)} &
\end{array}$$

So the 2-cycle(s) are the roots of $\Phi_2(x) = \lambda^2x^2 - \lambda x + (1-\lambda)$

or

$$x_\mp^* = \frac{\lambda \mp \sqrt{\lambda^2 - 4\lambda^2(1-\lambda)}}{2\lambda^2} = \frac{1 \mp \sqrt{4\lambda-3}}{2\lambda}.$$

The 2-cycle is <u>complex</u> for $\lambda < 3/4$; and <u>real</u> for $\lambda > 3/4$.

At $\lambda = 3/4$ the two points collide, degenerating into a 1-cycle, which coincides with the fixed point found above.

We investigate the stability of the 2-cycle. The multiplier at each of the two points is

$$f^2(x^*_\mp)' = f'(x^*_+) f'(x^*_-) = (-2\lambda x^*_+)(-2\lambda x^*_-)$$
$$= 4\lambda^2 x^*_+ x^*_-.$$

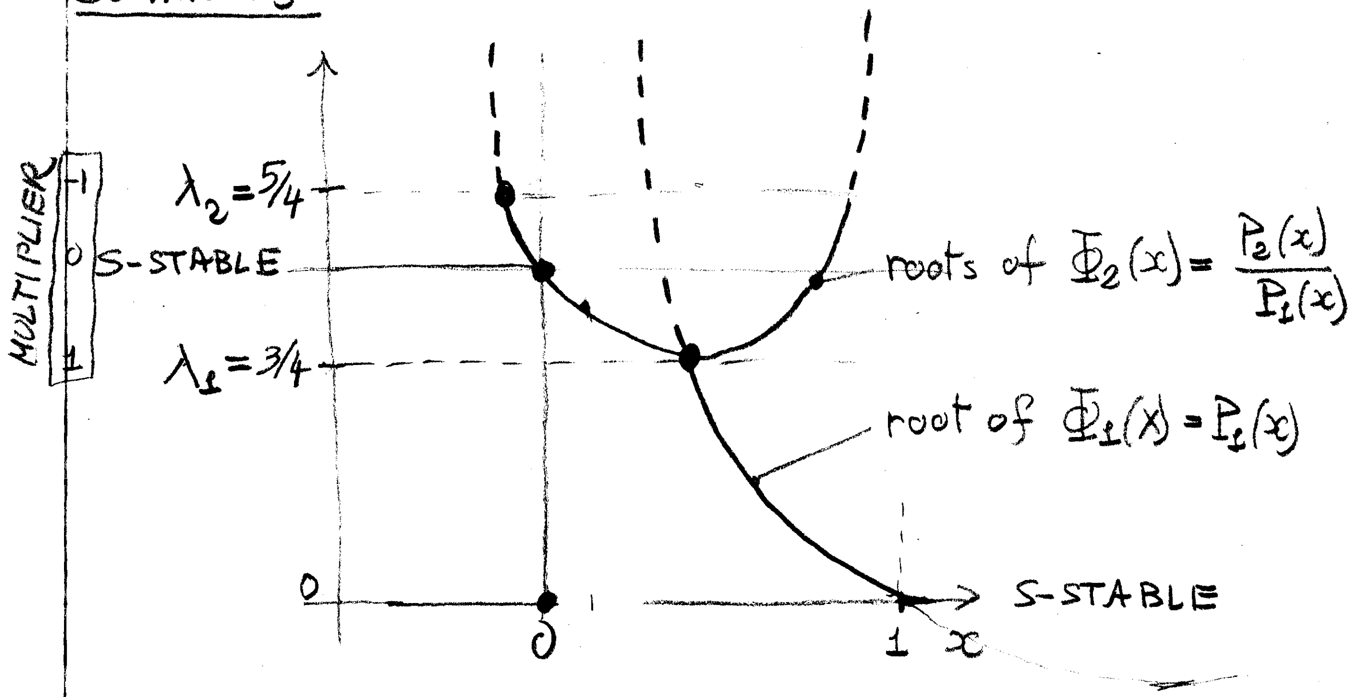
Now $\frac{1}{\lambda^2}\Phi_2(x) = x^2 - \frac{1}{\lambda}x + \frac{1-\lambda}{\lambda^2} = (x - x^*_+)(x - x^*_-)$ so the product of the roots (the constant term) is $(1-\lambda)/\lambda^2$, whence

$$f^2(x^*_\mp)' = 4\lambda^2 \frac{1-\lambda}{\lambda^2} = 4(1-\lambda).$$

At $\lambda = 3/4$ we have $f^{2'}(x^*_\alpha) = 1$. The multiplier decreases monotonically, becoming 0 at $\lambda = 1$ (the 2-cycle is superstable), and $= -1$ at $\lambda = 5/4$, beyond which the 2-cycle is unstable.

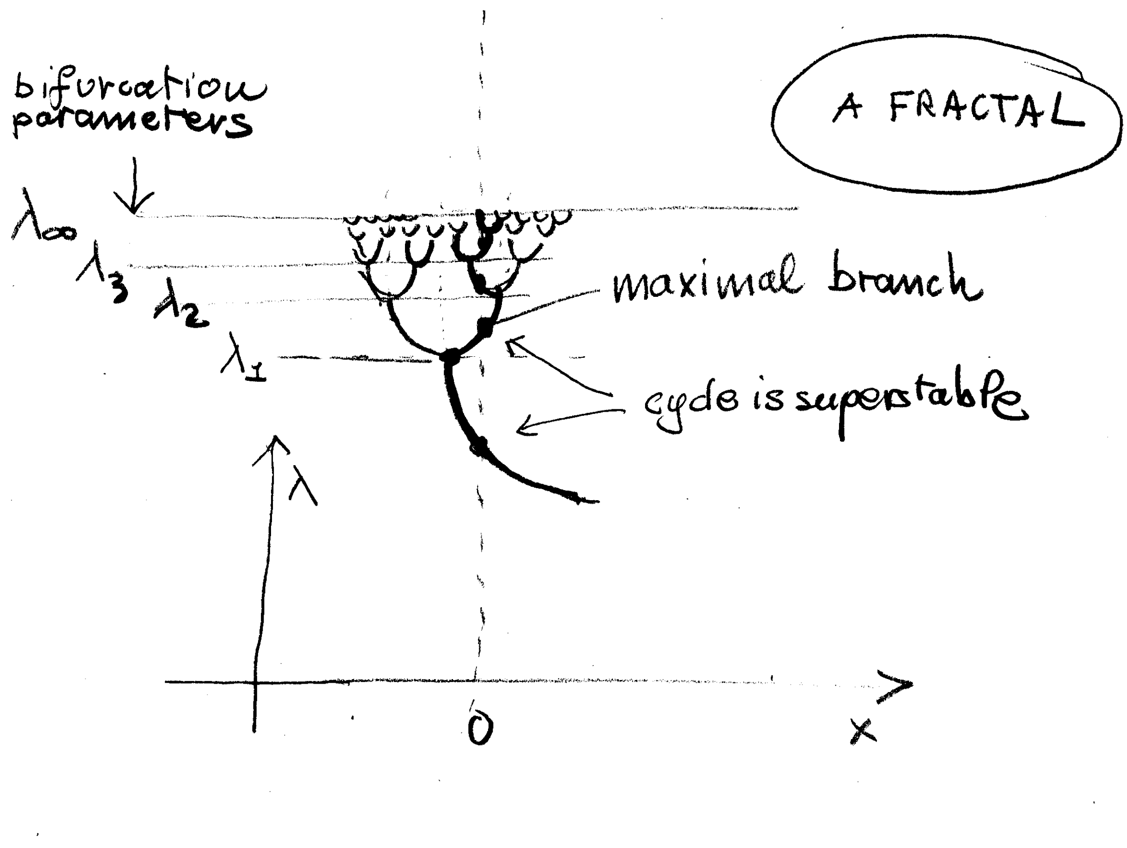
<u>Check</u>: at $\lambda = 1$ we have $x^*_- = 0 \quad x_+ = 1$.

Summary



The event occurring at $\lambda_1 = 3/4$ is called a <u>period-doubling</u> (or pitch-fork) <u>bifurcation</u>.

The scenario continues, resulting in a <u>bifurcation tree</u>



Thm Let $f_\lambda(x) = 1-\lambda x^2$. There exists a monotonically increasing sequence of parameter values

$$\lambda_1, \lambda_2, \lambda_3, \dots \quad = 3/4, 5/4, \dots\dots$$

with the property that for all $j \geq 1$:

A real 2^j-cycle is born at $\lambda=\lambda_j$, with multiplier 1. The multiplier decreases monotonically, to become -1 at $\lambda=\lambda_{j+1}$, resulting in loss of stability.

Furthermore, the above sequence has a limit

$$\lim_{j\to\infty} \lambda_j = \lambda_\infty = 1.401155189\dots \qquad (*)$$

which is approached geometrically, with

$$\lim_{j\to\infty} \frac{\lambda_j - \lambda_{j-1}}{\lambda_{j+1}-\lambda_j} = 4.6692016\dots = \delta \qquad (**)$$

'Feigenbaum constant' (1978).

Let $\bar{\lambda}_j$ with $\lambda_j < \bar{\lambda}_j < \lambda_{j+1}$ be the parameter value at which the 2^j-cycle is superstable.
Define

$$\alpha_j = f_{\bar{\lambda}_j}^{2^{j-1}}(0)$$

then

$$\lim_{j\to\infty} \frac{\alpha_j}{\alpha_{j+1}} = \alpha = -2.502907875095\dots$$

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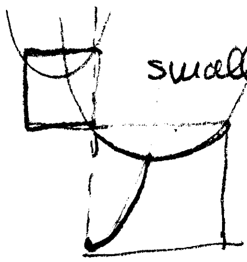
Remarks:

- In the limits (*) and (**) λ_j can be replaced by $\bar{\Lambda}_j$.
- The quantity α_j is the (signed) distance between the maximal branch and the nearest branch, at superstability.
  Thus δ & α describe the scaling properties of the bifurcation tree, in the λ-x space.



- Feigenbaum (1978) discovered that δ and α are universal constants: any map with a single quadratic maximum at $x = x_c$ has, under suitable parametrisation, period-doubling scenarios with the same constants!
  (The definition of α_j must be generalized to
  $$\alpha_j = f_{\Lambda_j}^{2^{j-1}}(x_c) - x_c.)$$

These claims were later proved by Lanford & Sullivan.

Ex $\quad f_\lambda(x) = \lambda \sin(x) \qquad x \in [0, \pi]$

(Find the first bifurcations numerically.)

computing superstable orbits

We illustrate the algorithm for the logistic map $f_\lambda(x) = 1 - \lambda x^2$, which has a unique critical point at zero: $f_\lambda'(0) = 0$. Such point must belong to any superstable T-cycle, which is therefore a solution to

$$f_\lambda^T(0) = 0$$

or, since $f(0) = 1$, $f_\lambda^{T-1}(1) = 0$.

$T = 1$ $\quad f_\lambda^1(0) = 1 - \lambda 0^2 = 1$, no solution (degen. for $\lambda = 0$)

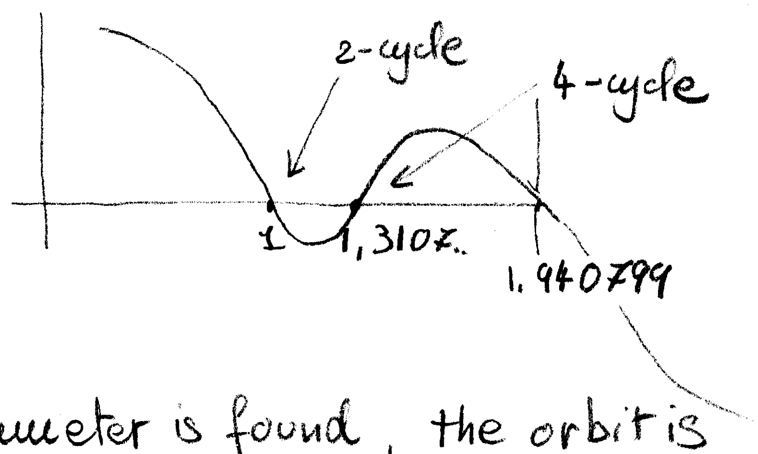
$T = 2$ $\quad f_\lambda^2(0) = f_\lambda(1) = 1 - \lambda \cdot 1^2 = 0 \quad \Rightarrow \quad \lambda = 1$

$T = 3$ $\quad f_\lambda^3(0) = f_\lambda^2(1) = 1 - \lambda(1 - \lambda \cdot 1^2)^2 = 1 - \lambda + 2\lambda^2 - \lambda^3$

This polynomial has a real root at $\lambda = 1.75487\ldots$. (we shall return to this λ-value below).

$T = 4$ $\quad f_\lambda^4(0) = f_\lambda^3(1) = 1 - \lambda\left(1 - \lambda(1-\lambda)^2\right)^2$

This polynomial has 3 real roots



Once the parameter is found, the orbit is easily computed as

$$0, f(0), f^2(0), \ldots, f^{T-1}(0)$$