Some general definitions

- $f: A \to B$ is bijective if it is both injective & surjective.
  In this case $f^{-1}: B \to A$ exists.

- $f: A \to B$ is continuous if $\lim_{x \to x_0} f(x) = f(x_0)$, $\forall x_0 \in A$.

- Let $f: \mathbb{R} \to \mathbb{R}$. $f \in C^r$ if f has r continuous derivatives.

- $f: \mathbb{R} \to \mathbb{R}$ is a homeomorphism if it is bijective and if f, f^{-1} are both continuous.
  If $f, f^{-1} \in C^r$ then we say that f is a C^r-diffeomorphism.

  A C^1-diffeo is just called a diffeo.
  A C^0-diffeo is a homeo.

Example

$f(x) = x^3$ is a C^∞ map $\mathbb{R} \to \mathbb{R}$, but it is only a C^0-diffeo, as f^{-1} is not differentiable at zero.

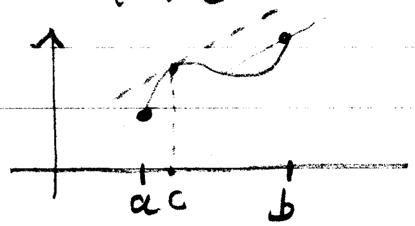
Example

$f(x) = 1 - \lambda x^2$ is not even a homeo, since it is not invertible.

Mean value theorem (MVT)

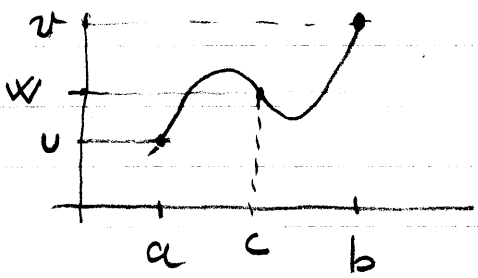
If $f:[a,b]\to\mathbb{R}$ is C^1 then $\exists\, c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$



Intermediate value theorem (IVT)

If $f:[a,b]\to\mathbb{R}$ is continuous and $f(a)=u$, $f(b)=v$, then given any $w\in(u,v)$, $\exists\, c\in(a,b)$ such that $f(c)=w$



(c may not be unique!)

Lemma Let $I=[a,b]\subset\mathbb{R}$. Any continuous $f:I\to I$ has a fixed point.

Pf: Define $\Phi(x)=f(x)-x$. Since $f:I\to I$, we have

$$f(a)\geq a \quad\Rightarrow\quad \Phi(a)\geq 0$$
$$f(b)\leq b \quad\Rightarrow\quad \Phi(b)\leq 0$$

g is continuous, so IVT $\Rightarrow\ \exists\, c\in(a,b)$ with $\Phi(c)=0$, i.e. $f(c)=c$, and c is the desired fixed point

$\square$

Diffeomorphisms of $\mathbb{R}$

let $f: \mathbb{R} \to \mathbb{R}$ be a diffeo. Its inverse f^{-1} is defined by

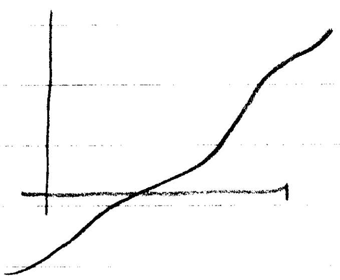
$$y = f(x) \iff x = f^{-1}(y)$$

Thus $f^{-1}(f(x)) = x$, and from the chain rule

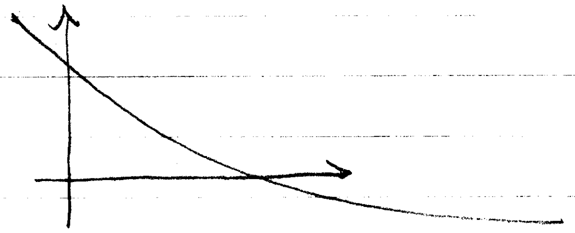
$$(f^{-1})'(f(x)) \cdot f'(x) = 1 \quad \text{or} \quad (f^{-1})'(y) = 1/f'(x)$$

For a diffeo. we must have $f'(x) \neq 0$ (lest $(f^{-1})'(y)$ is infinite!), and therefore, since f is continuous IVT implies that $f'(x)$ is either positive or negative for all x.

In other words, a diffeo $f: \mathbb{R} \to \mathbb{R}$ is either
<u>order preserving</u> $(x < x' \Rightarrow f(x) < f(x'))$ or
<u>order reversing</u> $(x < x' \Rightarrow f(x) > f(x'))$ (MVT)



order-preserving



order-reversing

<u>Theorem 1.1</u> If f is an <u>order-reversing</u> diffeo of $\mathbb{R}$, then f has exactly one fixed point.

<u>Pf:</u> Let $B = \lim_{x \to -\infty} f(x)$ and $A = \lim_{x \to \infty} f(x)$ (A, B, could be $\mp\infty$). Then $B > A$ since f reverses the order. Now let $\Phi(x) = f(x) - x$ so that

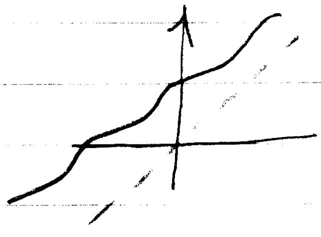
$$\lim_{x \to -\infty} \Phi(x) = -\infty \qquad \lim_{x \to \infty} \Phi(x) = -\infty$$

and therefore $\exists c$ with $\Phi(c) = 0 \Rightarrow f(c) = c$.

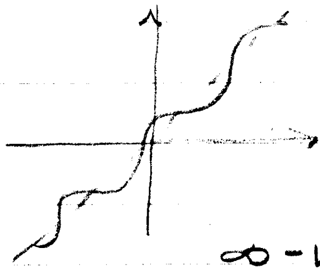
To prove uniqueness, suppose $f(c) = c$, $f(d) = d$. with $c < d$, say. Then $f(c) > f(d)$ since f is order-reversing. But since c & d are fixed points, this implies $c > d$, a contradiction.

$\square$

__Remark__ If f is order-preserving, there may be any number of fixed points.

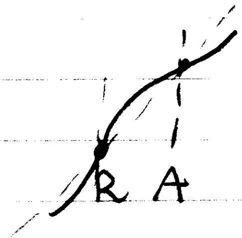


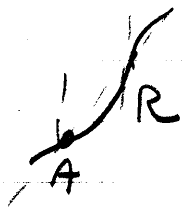
none



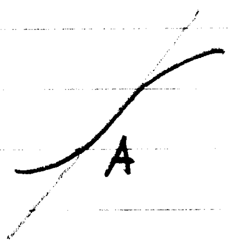
∞-many

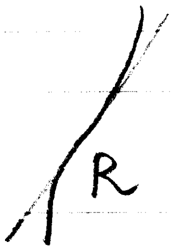
If their multiplier is $\neq 1$, they must occur in an alternating attracting-repelling sequence

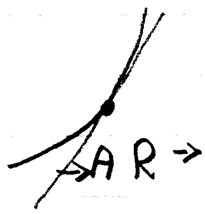




By labeling the marginal cases as follows







then A is always followed by R, & vice-versa

Thm 1.2 An order-preserving diffeo $f: \mathbb{R} \to \mathbb{R}$ has no periodic orbits of period $n > 1$.

Pf: Let $\{x_0, x_1 = f(x_0), \ldots\}$ be the orbit of x_0, and let $f(x_0) \neq x_0$.

Case 1 $\quad x_1 > x_0 \implies f(x_1) > f(x_0) \iff x_2 > x_1$
repeating the argument, we get
$$x_0 < x_1 < x_2 < \cdots < x_n \implies x_n \neq x_0.$$

Case 2 $\quad x_1 < x_0 \implies f(x_1) < f(x_0) \iff x_2 < x_1$
thus
$$x_0 > x_1 > \cdots > x_n. \quad \text{Hence } x_n \neq x_0. \quad \square$$

What periods can we have in the order-reversing case? We have 1 fixed point, and we can have any number of 2-cycles (eg; $f(x) = -x$).
However,

Thm 1.3 An order-reversing diffeo $f: \mathbb{R} \to \mathbb{R}$ has no periodic orbit of period $n > 2$.

Pf: f diffeo with $f'(x) < 0 \implies$
f^2 diffeo with $(f^2)'(x) > 0$.

Indeed f^2 has inverse f^{-2} and
$$f^{2\prime}(x) = f'(f(x)) f'(x) > 0.$$

So f^2 has no n-cycles with $n > 1$, whence f has no $2n$-cycles with $n > 1$.

If n is odd, then f^n is a diffeo with $f^{n'}(x) < 0$, and so f^n has a unique fixed point (thm 3.1), which is necessarily the fixed point of f. □

Summary Diffeo $f: \mathbb{R} \to \mathbb{R}$

	fixed pts	2-cycles	n-cycles, $n > 2$
$f'(x) > 0$	arbitrary	none	none
$f'(x) < 0$	$\exists$ unique	arbitrary	none

The behaviour can be much more complicated if

- we relax the condition of bijectivity
- we increase the dimension.