I. Continuous & discrete time dynamical systems

<u>Def</u> A continuous-time dynamical system on $\mathbb{R}^N$ is a set of first order ODE's of the form

$$\begin{cases} \dot{x}^{(1)} = v_1(x^{(1)}, \dots, x^{(N)}) \\ \dot{x}^{(2)} = v_2(x^{(1)}, \dots, x^{(N)}) \\ \vdots \\ \dot{x}^{(3)} = v_N(x^{(1)}, \dots, x^{(N)}) \end{cases} \qquad (1)$$

abbreviation: $\dot{X} = V(X)$.

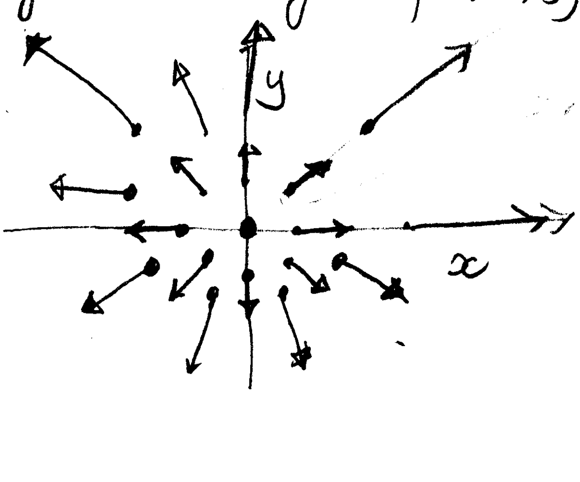
where $X = (x^{(1)}, \dots, x^{(N)}) \in \mathbb{R}^N$, $\dot{x}^{(k)} = \frac{d}{dt} x^{(k)}$,

$V = (v_1, \dots, v_N)$ is an array of <u>differentiable</u> functions $v_k : \mathbb{R}^N \to \mathbb{R}$ so that $V : \mathbb{R}^N \circlearrowleft$.
$\mathbb{R}^N$ is called the <u>phase space</u> of the system (1).

We often write (x, y) for $(x^{(1)}, x^{(2)})$, etc.

<u>Ex</u> $\begin{cases} \dot{x} = x = v_1(x,y) \\ \dot{y} = y = v_2(x,y) \end{cases}$ $\quad V(x,y) = (x, y) \in \mathbb{R}^2$.

At each point of $\mathbb{R}^2$ there is a vector $\vec{v} = V(x,y)$ describing the change of (x,y) with time



a vector <u>field</u> on the plane.

Ex From Newton's law of motion to a vector field

$$F = ma \qquad \text{1-D:} \quad F = F(x); \quad a = \ddot{x} = \frac{d^2x}{dt^2}$$

Rewrite $\ddot{x} = \frac{1}{m} F(x)$ 2nd order ODE

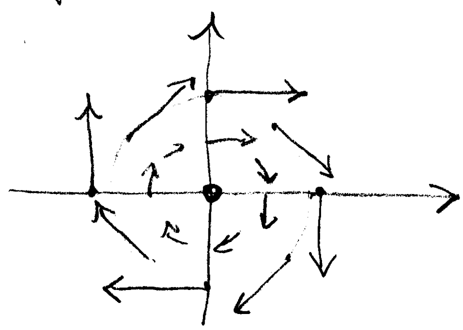
Transform into <u>two</u> ODEs of 1st order:

Let $y = \dot{x}$ then $\dot{y} = \ddot{x}$ and $F = ma$ becomes

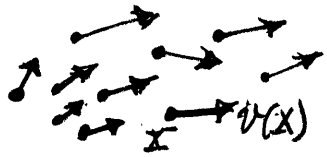
$$\begin{cases} \dot{x} = y = v_1(x,y) \\ \dot{y} = \frac{1}{m} F(x) = v_2(x,y). \end{cases}$$

Let $m = 1$ and $F(x) = -x$ (simple harmonic motion)

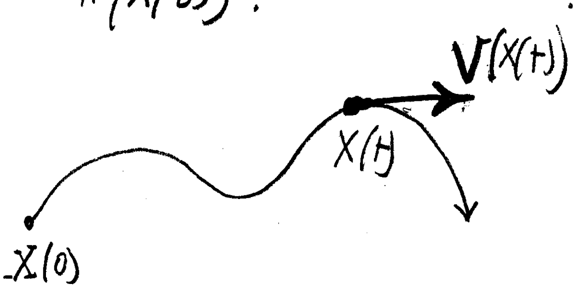
then $\vec{v}(x,y) = (y, -x)$



A dynamical system of the type (1) defines a *vector field* $V = V(X)$ on $\mathbb{R}^N$.



Solving the ODE (1) with initial conditions $X_0 = (x_0^{(1)}, \ldots, x_0^{(N)})$ means finding a differentiable curve $t \mapsto X(t)$ in $\mathbb{R}^N$ with the property that $X(0) = X_0$ & $\dot{X}(t) = V(X(t))$.



It can be proved that the vector $V(X(t))$ is tangent to the curve at $X(t)$. Such a curve is called the *orbit* through X_0.

Ex $\begin{cases} \dot{x} \\ \dot{y} \end{cases} = \begin{matrix} ax \\ ay \end{matrix}$ with i.c. (x_0, y_0)

solution: $\frac{dx}{dt} = ax \Rightarrow \int \frac{dx}{x} = \int a\,dt \Rightarrow$

$\Rightarrow \ln x = at + c \qquad x(t) = e^{at+c} = e^c e^{at}$

Let c be s.t. $e^c = x_0$. Then $x(t) = x_0 e^{at}$, and similarly $y(t) = y_0 e^{at}$, whence $X(t) = (x_0 e^{at}, y_0 e^{at})$ is the orbit (a ray).

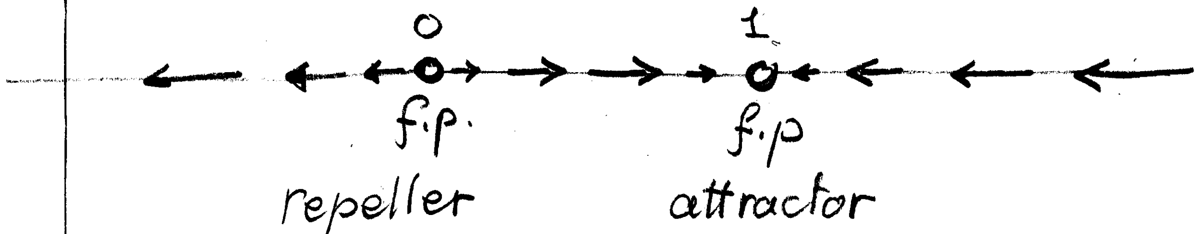
Remarks:

1. Vector fields can be defined on any coordinate space (a "manifold"), not just on $\mathbb{R}^N$.
2. If the vector field is smooth (i.e., all partial derivatives of all v_k exist and are continuous), then the orbit through X_0 is unique (we shall not prove this).

If $v(X_0) = 0$, then $X(t) = X_0$, and we speak of a <u>fixed point</u>. A fixed point is characterized by the behaviour of nearby orbits.

<u>Ex</u> 1-D $\qquad \dot{x} = x(1-x) = v(x)$

$$v(x): \begin{cases} = 0 & \text{for } x = 0, 1 \\ > 0 & \text{for } 0 < x < 1 \\ < 0 & \text{otherwise.} \end{cases}$$

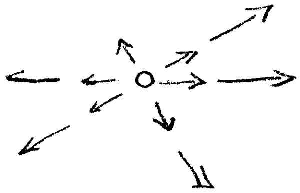


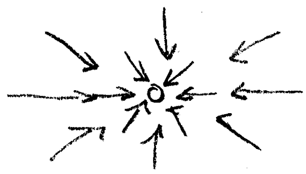
We have determined the qualitative behaviour of the solutions, without solving the equation! It turns out that the solution is

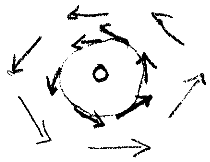
$$x(t) = \frac{x_0 e^t}{x_0 e^t + 1 - x_0} \qquad \text{(check!)}$$

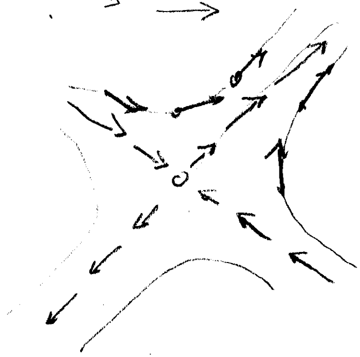
Since v is differentiable, such solution is unique.

Ex The following vector fields on $\mathbb{R}^2$ have a fixed point at the origin

$v(x,y) = (x,y)$  Repeller

$v(x,y) = (-x,-y)$  attractor

$v(x,y) = (-y, x)$  centre (elliptic)

$v(x,y) = (y, x)$  saddle (hyperbolic)

Other possibilities  , etc.

In dimension 3 or higher, the behaviour can be exceedingly complicated

Ex The Lorenz model (1963)

$$\dot{x} = \sigma x - \sigma y$$
$$\dot{y} = rx - y - x^2 z$$
$$\dot{z} = xy - bz$$

σ, r, b parameters

$\sigma = 10$
$b = 2.67$
$r = 28$

From continuous to discrete time

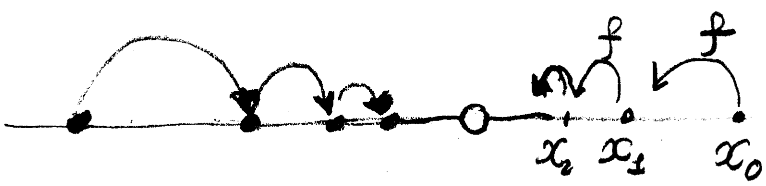
Consider again the dynamical system $\dot{x} = ax$, with solution $x(t) = x_0 e^{at}$.
We examine the solution at integer values n of the time t, writing x_n for $x(n)$.

$$x_n = x_0 e^{an}$$

$$x_{n+1} = x_0 e^{a(n+1)} = e^{a} x_n$$

Letting $\lambda = e^{a}$, we have the "stroboscopic map"

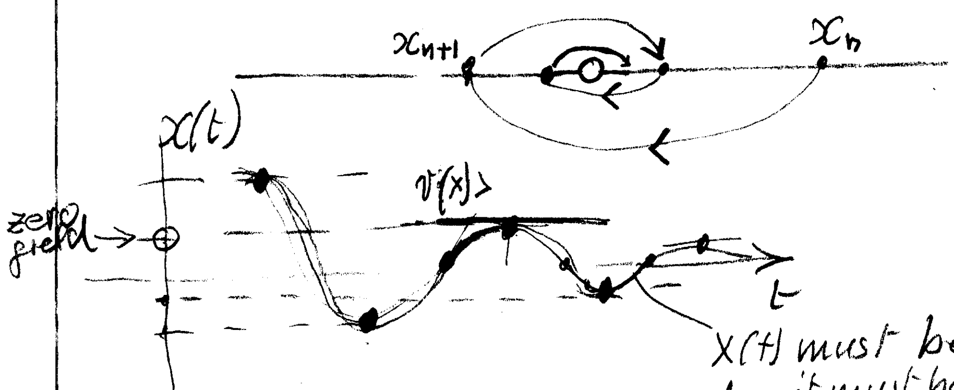
$$x_{n+1} = f_\lambda(x_n) = \lambda x_n$$



$0 < \lambda < 1$
or $a < 0$

Transforming a continuous-time dynamical system into a discrete-time one is always possible in principle, although often impossible in practice. The converse process is more subtle.

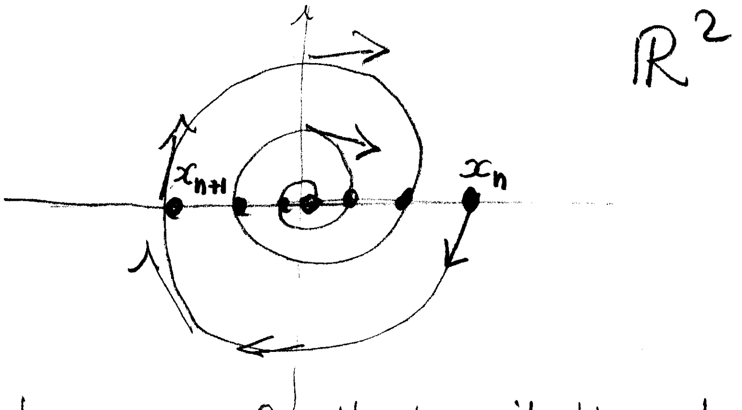
Ex $\quad x_{n+1} = -\frac{1}{2} x_n \qquad$ Can we find a vector field?



NOT in $\mathbb{R}$!

x(t) must be differentiable, and so it must have ∞ many stationary pts.

We must increase the dimension by 1 :



In the above example, the transit time between intersection of an orbit with the x-axis need not be unity, in which case the index n simply counts such intersection. This construct is called a <u>surface of section</u>