EXPANSIONS IN NON-INTEGER BASES

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1. Introduction into β-expansions

Representations of real numbers in non-integer bases were introduced by Rényi [17] and first studied by Rényi and by Parry [16].

Let first β be an integer greater than 1. Then any number $x \in [0, 1)$ can be represented in the form

$$x = \sum_{n=1}^{\infty} a_n \beta^{-n}, \qquad a_n \in \{0, 1, \dots, \beta\}.$$

This representation is unique, except for a countable set of x. The corresponding map here is $\tau_\beta : [0, 1) \to [0, 1)$ defined by the formula

$$\tau_\beta(x) = \beta x \bmod 1.$$

This map acts as the shift on the expansions, i.e., $a_n(\tau_\beta x) = a_{n+1}(x)$. The properties of this map are well known; in particular, it preserves the Lebesgue measure on the interval, and the corresponding dynamical system has various nice properties. See Figure 1 for the case $\beta = 2$.

Assume now $\beta > 1$ to be non-integer. We call any representation of the form

$$x = \sum_{n=1}^{\infty} a_n \beta^{-n}, \qquad a_n \in \{0, 1, \dots, \lfloor \beta \rfloor - 1\}.$$

a *β-expansion of* x. (Here $\lfloor t \rfloor$ denotes the integer part of t.) For instance, for $\beta \in (1, 2)$ – which is going to be our main example – the set of "digits" is $\{0, 1\}$, i.e., like the one for the binary expansions. It is easy to show "by hand" that any $x \in \left[0, \frac{\lfloor \beta \rfloor}{\beta - 1}\right]$ has at least one β-expansion.

We will do it in a way similar to the standard doubling map. Let us assume for simplicity that $1 < \beta < 2$ and introduce the following *multivalued map*:

$$T_\beta(x) = \begin{cases} \beta x, & x \in \left[0, \frac{1}{\beta}\right] \\ \beta x \ \text{ or } \ \beta x - 1, & x \in \left(\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}\right) \\ \beta x - 1, & x \in \left[\frac{1}{\beta(\beta-1)}, \frac{1}{\beta-1}\right] \end{cases}$$

(see Figure 2).

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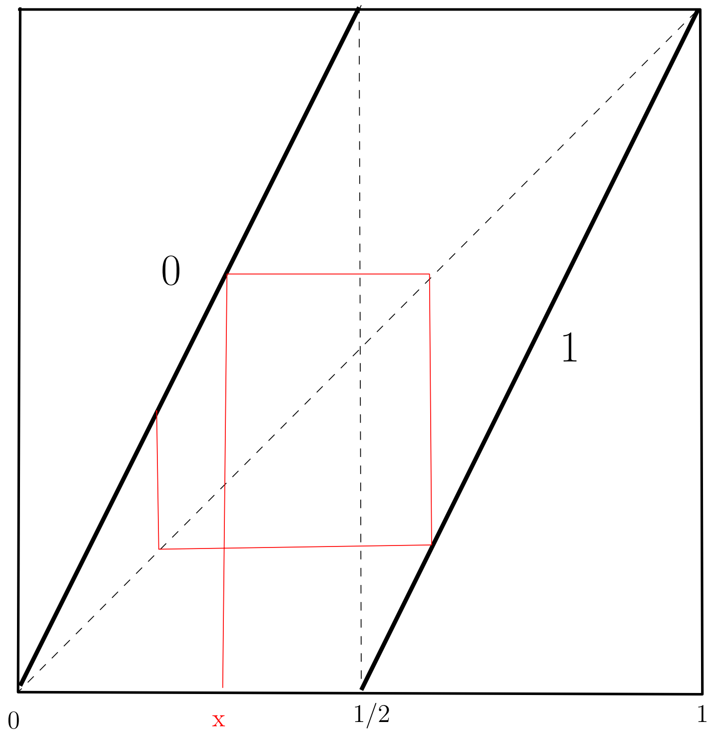

FIGURE 1. The doubling map

We see that if $x \in \left[0, \frac{1}{\beta}\right)$ or $x \in \left(\frac{1}{\beta(\beta-1)}, \frac{1}{\beta-1}\right]$, then $T_\beta(x)$ is uniquely defined. However, whenever x lies in the *switch region* $\left[\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}\right]$, we have a choice between 0 and 1.

Figure 3 depicts a branching pattern that occurs for the multivalued map T_β. We will see that typically it is indeed a binary tree.

If we always choose 1 (or, in the general case, the largest possible "digit"), such an expansion is called *greedy*. The map T_β becomes the *β-transformation* $\tau_\beta x = \beta x \bmod 1$ (restricted to $[0, 1)$) – see Figure 4.

Although τ_β does not preserve the Lebesgue measure, there exists a bounded positive density function h_β such that the absolutely continuous measure μ_β given by h_β is τ_β-invariant (see [16]). The dynamical system $([0, 1), \mu_\beta, \tau_\beta)$ is well studied, and its properties are similar to the ones of the doubling map.

Theorem 1. *([9]) If $\beta < \frac{1+\sqrt{5}}{2}$, then any $x \in (0, 1/(\beta - 1))$ has a continuum of distinct β-expansions.*

Proof. One can check (exercise!) that if $x < 1/\beta$, then it is impossible that $T_\beta(x) > 1/(\beta(\beta - 1))$ – see Figure 5. Hence eventually the trajectory of any point bifurcates, and the procedure repeats for each of the images, ad infinitum. □

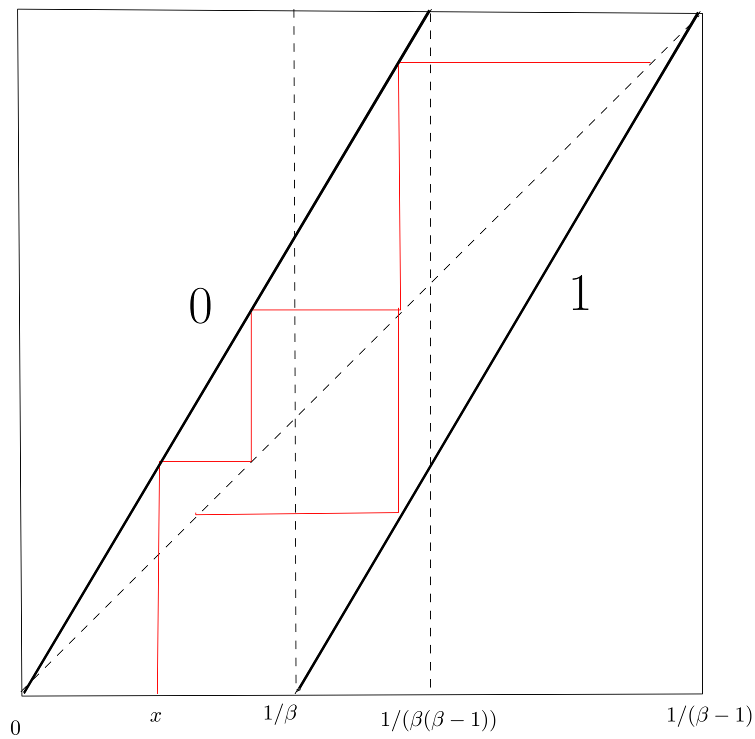

FIGURE 2. Multivalued β-transformation T_β

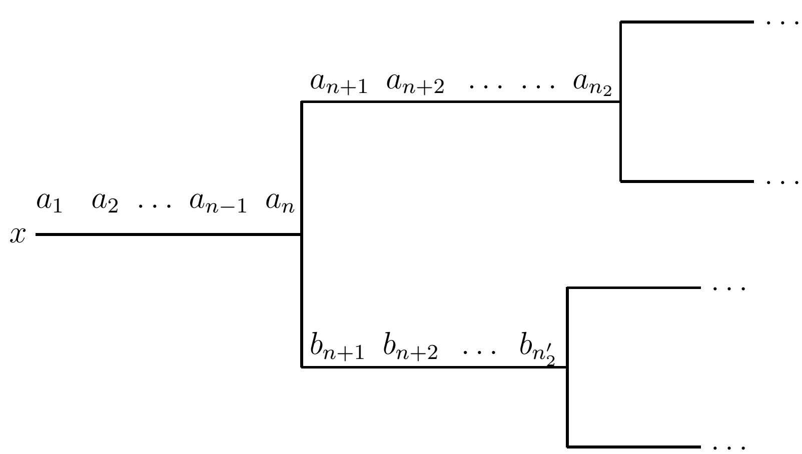

FIGURE 3. Branching and bifurcations

A quantitative version of this result has been recently proven by Feng and the author. Put

$$\mathcal{N}_n(x;\beta) = \#\left\{(a_1,\dots,a_n)\in\{0,1\}^n \mid \exists(a_{n+1},a_{n+2},\dots) : x=\sum_{k=1}^{\infty} a_k\beta^{-k}\right\}.$$

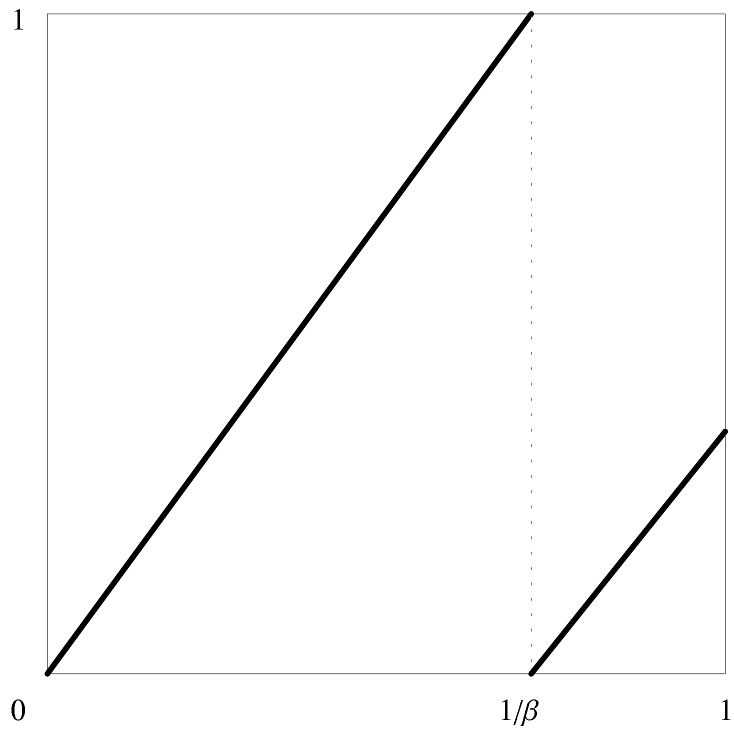

FIGURE 4. The β-transformation τ_β

Theorem 2. *([12]) Let β be an arbitrary number in $\left(1, \frac{1+\sqrt{5}}{2}\right)$. Then there exists $c = c(\beta) > 0$ such that*

$$\liminf_{n\to\infty} \frac{\log \mathcal{N}_n(x;\beta)}{n} \geq c \quad \textit{for any } x \in \left(0, \frac{1}{\beta-1}\right).$$

What about when β is greater than the golden ratio? In this case one can show (exercise!) that there exists a point $x = x(\beta) < 1/\beta$ such that $T_\beta(x) > 1/(\beta(\beta-1))$, and $T_\beta^2(x) = x$ (a 2-cycle) – see Figure 6.

Hence the β-expansion of such a point is necessarily $010101\ldots$ We will discuss unique β-expansions in detail in the next section.

Thus, it is not true that every internal point has a continuum of β-expansions if β is between the golden ratio and 2. However, a weaker result is still valid:

Theorem 3. *(Sidorov [18, 19])*

(1) *Almost every point $x \in (0, 1/(\beta-1))$ has a continuum of β-expansions.*
(2) *Furthermore, the set of exceptions has Hausdorff dimension strictly less than 1.*

Proof. We will prove the first part. Our first goal is to show that a.e. $x \in (0,1)$ has at least two different β-expansions. We may assume that $\beta \geq \frac{1+\sqrt{5}}{2}$.

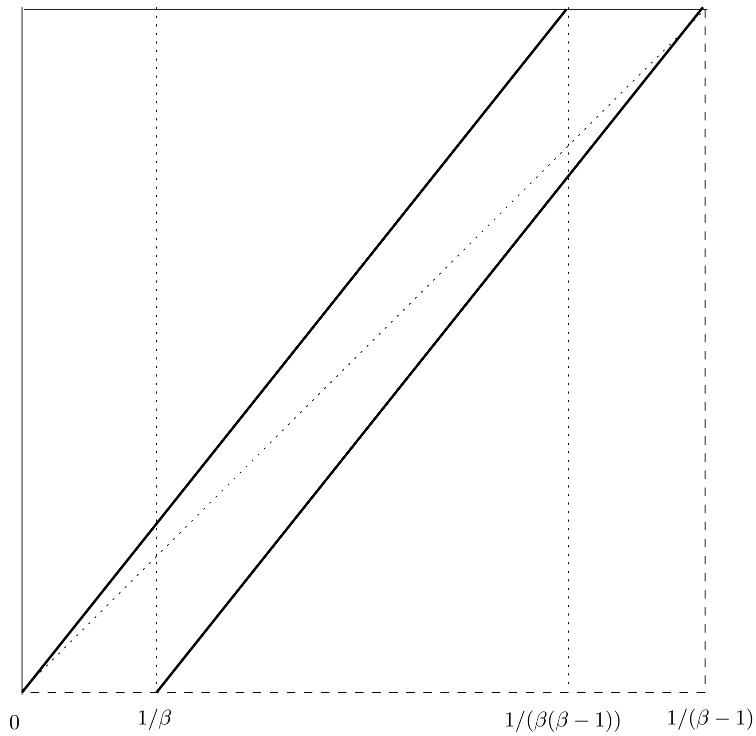

FIGURE 5. The β-transformation T_β for $\beta = 1.25$

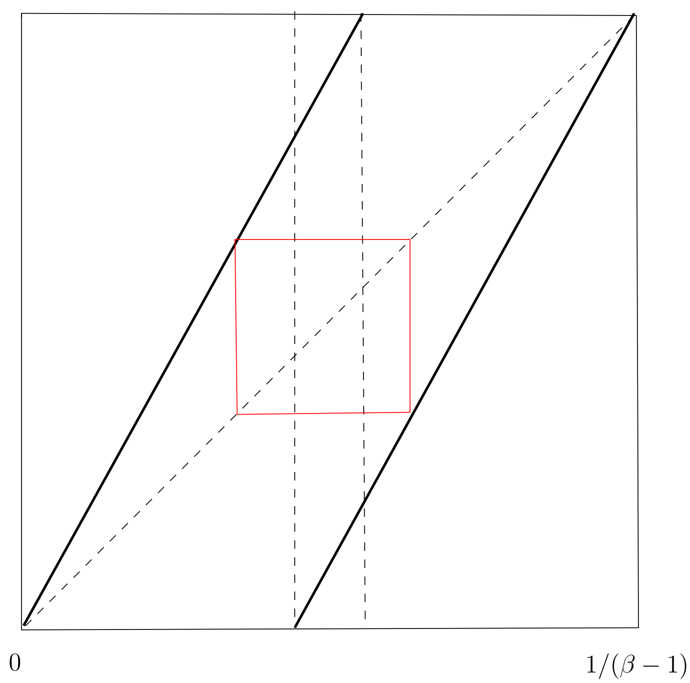

FIGURE 6. The 2-cycle

Since β belongs to $[(1+\sqrt{5})/2, 2)$, there exists $m = m(\beta) \ge 2$ such that

$$1 + \beta^{-m+1} < \frac{1}{\beta - 1}; \tag{1.1}$$

specifically, we can take

$$m = \left\lfloor \log_\beta \frac{\beta - 1}{2 - \beta} \right\rfloor + 1 \ge 2$$

(for $\beta = (1+\sqrt{5})/2$ we have $\beta - 1 = \beta^{-1}, 2 - \beta = \beta^{-2}$, whence $\log_\beta \frac{\beta-1}{2-\beta} = 1$).

So, we consider x in $(0, 1)$, and assume that its greedy expansion is of the form

$$(\varepsilon_1, \dots, \varepsilon_n, 1, \underbrace{0, \dots, 0}_{m-1}, \varepsilon_{n+m+1}, \dots).$$

We can construct a different β-expansion for x. Namely, if $x' = \sum_{j=1}^n \varepsilon_j \beta^{-j}$, then

$$x - x' = \beta^{-n-1} + \sum_{j=n+m+1}^{\infty} \varepsilon_j \beta^{-j} \in [\beta^{-n-1}, \beta^{-n-1} + \beta^{-n-m}],$$

because $\sum_{n+m+1}^{\infty} \varepsilon_j \beta^{-j} \le \beta^{-n-m}$ (a property of the greedy expansions). On the other hand, we infer from (1.1) that

$$\beta^{-n-1} + \beta^{-n-m} < \beta^{-n-2} + \beta^{-n-3} + \dots = \frac{\beta^{-n-1}}{\beta - 1},$$

whence

$$x - x' < \beta^{-n-2} + \beta^{-n-3} + \cdots$$

as well. This means that if we put $\varepsilon'_{n+1} = 0$, it is possible to find $(\varepsilon'_{n+2}, \varepsilon'_{n+3}, \dots)$ in Σ such that $x = \sum_{j=1}^{\infty} \varepsilon'_j \beta^{-j}$. By our construction, $\varepsilon_{n+1} \ne \varepsilon'_{n+1}$.

Thus, the set $\mathcal{U}_\beta$ – all x which have a unique β-expansion – has measure zero. Now, if for some x its tree of β-expansions (see Figure 3) is not the full binary tree, it means that one of the branches "flatlines". This implies that for one of β-expansions of x, say, for $(\varepsilon_1, \varepsilon_2, \dots)$, there exists k such that $(\varepsilon_k, \varepsilon_{k+1}, \dots)$ is a unique expansion (since it does not bifurcates any further).

Since any shift of a β-expansion is either βx or $\beta x - 1$, we infer that x belongs to a scaled copy of $\mathcal{U}_\beta$. Any such copy has zero measure and there is only a countable set of them for x to lie in. Hence the set of x whose branching is not full is a zero measure set. In particular, a.e. x has a continuum of β-expansions. □

Finally, we would like to mention random β-expansions. Again, we assume for simplicity that $1 < \beta < 2$. Put $\Omega = \{0, 1\}^{\mathbb{N}}$, and we regard 0 as "tails" and 1 as "heads". We introduce the *random β-transformation* $K_\beta : \left[0, \frac{1}{\beta-1}\right] \times \Omega \to \left[0, \frac{1}{\beta-1}\right] \times \Omega$ as follows:

$$K_\beta(x, \omega) = \begin{cases} (\beta x, \omega), & x \in \left[0, \frac{1}{\beta}\right) \\ (\beta x - \omega_1, \sigma(\omega)), & x \in \left[\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}\right] \\ (\beta x - 1, \omega), & x \in \left(\frac{1}{\beta(\beta-1)}, \frac{1}{\beta-1}\right] \end{cases}$$

Here $\sigma : \Omega \to \Omega$ is the one-sided shift, i.e., $\sigma(\omega_1, \omega_2, \omega_3, \dots) = (\omega_2, \omega_3, \dots)$. In other words, if we are outside the switch region, we just apply βx or $\beta x - 1$ respectively and do not touch the "coin". If we are in the switch region, we flip a coin (= check ω_1) and apply the corresponding map, after which we shift ω for the next flip, whenever we'll need it.

It has been shown in [4] that there exists a unique probability measure m_β on $\left[0, \frac{1}{\beta-1}\right]$ such that m_β is equivalent to the Lebesgue measure and $m_\beta \otimes \mathbb{P}$ is invariant and ergodic under K_β, where $\mathbb{P} = \prod_1^\infty \left\{\frac{1}{2}, \frac{1}{2}\right\}$.

2. Unique β-expansions and their dynamics

Let, as above, $\mathcal{U}_\beta$ denote the set of $x \in (0, 1/(\beta-1))$ which have a unique β-expansion. Put $G = \frac{1+\sqrt{5}}{2}$.

Theorem 4 (Glendinning-Sidorov, 2001 [14])**.** *We have the following dichotomy:*

- *The set $\mathcal{U}_\beta$ is infinite countable if $\beta \in (G, \beta')$, and each unique expansion is eventually periodic.*
- *If $\beta \in (\beta', 2)$, then $\mathcal{U}_\beta$ has the cardinality of the continuum and a positive Hausdorff dimension.*

Here β' is the Komornik-Loreti constant which is defined as follows: denote by

$$(\mathfrak{m}_k)_{k=0}^\infty = 0110\ 1001\ 0110\ 1001\dots$$

the *Thue-Morse sequence*, i.e., the fixed point of the substitution $0 \to 01, 1 \to 10$.

The *Komornik-Loreti constant* $\beta' \approx 1.78723$ is defined as the unique solution of the equation

$$\sum_{k=1}^\infty \mathfrak{m}_k x^{-k} = 1.$$

This constant proves to be the smallest β such that $1 \in \mathcal{U}_\beta$. Allouche and Cosnard [2] have proved that β' is transcendental.

The topology of $\mathcal{U}_\beta$ can be complicated, depending on β. For some β it is a Cantor set, for some it isn't. For more detail see [15].

The set $\mathcal{U}_\beta$ is invariant under T_β (why?), hence we can consider $F_\beta = T_\beta|_{\mathcal{U}_\beta}$. Recall the *Sharkovskiĭ order* on $\mathbb{N}$:

$$\begin{array}{ccccccccccccc}
 & 3 & \rhd & 5 & \rhd & 7 & \rhd & \cdots & \rhd & 2m+1 & \rhd & \cdots \\
\rhd & 2\cdot 3 & \rhd & 2\cdot 5 & \rhd & 2\cdot 7 & \rhd & \cdots & \rhd & 2\cdot(2m+1) & \rhd & \cdots \\
\rhd & 4\cdot 3 & \rhd & 4\cdot 5 & \rhd & 4\cdot 7 & \rhd & \cdots & \rhd & 4\cdot(2m+1) & \rhd & \cdots \\
 & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
\rhd & 2^n\cdot 3 & \rhd & 2^n\cdot 5 & \rhd & 2^n\cdot 7 & \rhd & \cdots & \rhd & 2^n\cdot(2m+1) & \rhd & \cdots \\
 & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
 & & & \cdots & \rhd & 8 & \rhd & 4 & \rhd & 2 & \rhd & 1,
\end{array}$$

where the relation $a \rhd b$ indicates that a comes before b in the ordering.

Theorem 5 (**(Sharkovskiĭ's Theorem)**, see [5]). *Let f be a continuous automorphism of a compact interval I. If $k \rhd l$ in Sharkovkiĭ's ordering and if f has a point of smallest period k, then f also has a point of smallest period l.*

Now we are ready to state the main theorem of the this section. Put

$$U_n = \{\beta \in (1,2) : F_\beta \text{ has an } n\text{-cycle}\}.$$

(By the result quoted above, $U_2 = (G, 2)$, for instance.)

Theorem 6. *There exist real numbers β_n in $(1,2)$ such that $U_n = (\beta_n, 2)$ for any $n \ge 2$. Furthermore, $\beta_n < \beta_m$ if and only if $n \lhd m$ in the sense of the Sharkovskiĭ ordering.*

For a proof see [1]. Thus, once an n-cycle occurs at some β, it lives for any larger β. We have

$$G = \beta_2 < \beta_4 < \beta_8 < \cdots < \beta' < \cdots < \beta_7 < \beta_5 < \beta_3.$$

There exists an explicit formula for the minimal polynomial for β_n for any natural $n \ge 2$ (written as $n = 2^k(2\ell + 1)$) – see [1]. For the table of the first 8 values of β_n see Table 2.1 below.

β_n	period	minimal polynomial	numerical value	below β'?
$n = 2$	01	$x^2 - x - 1$	1.61803	yes
$n = 4$	0110	$x^3 - 2x^2 + x - 1$	1.75488	yes
$n = 8$	0110 1001	$x^5 - 2x^4 + x^2 - 1$	1.78460	yes
$n = 6$	011010	$x^6 - x^5 - x^4 - x^2 - 1$	1.78854	no
$n = 7$	0110101	$x^6 - 2x^5 + x^4 - x^3 - 1$	1.80509	no
$n = 5$	01101	$x^5 - x^4 - x^3 - x - 1$	1.81240	no
$n = 3$	011	$x^3 - x^2 - x - 1$	1.83929	no

TABLE 2.1. The table of β_n for small values of n

Figure 7 indicates how this problem can be related to the classical one-dimensional setting.

More precisely, define the map $h : \{0,1\}^{\mathbb{N}} \to \{L, R\}^{\mathbb{N}}$ as follows ($*$ denotes an arbitrary – but fixed – tail):

- $h(0*) = Lh(*)$;
- $h(1^a 0^b 1*) = RL^{a-1}RL^{b-1}h(1*)$ for $a, b \ge 1$;
- $h(1^a 0^\infty) = RL^{a-1}RL^\infty$;
- $h(1^\infty) = RL^\infty$.

Then h is one-to-one and maps the orbits of the shift on the set of unique β-expansions into the orbits of T_β which do not fall into C.

Let $\prec$ denote the standard *lexicographic order* on the sequences of 0s and 1s, namely, $\varepsilon \prec \varepsilon'$ if $\varepsilon_i \equiv \varepsilon_i',\ 1 \le i \le k$ and $\varepsilon_{k+1} < \varepsilon_{k+1}'$.

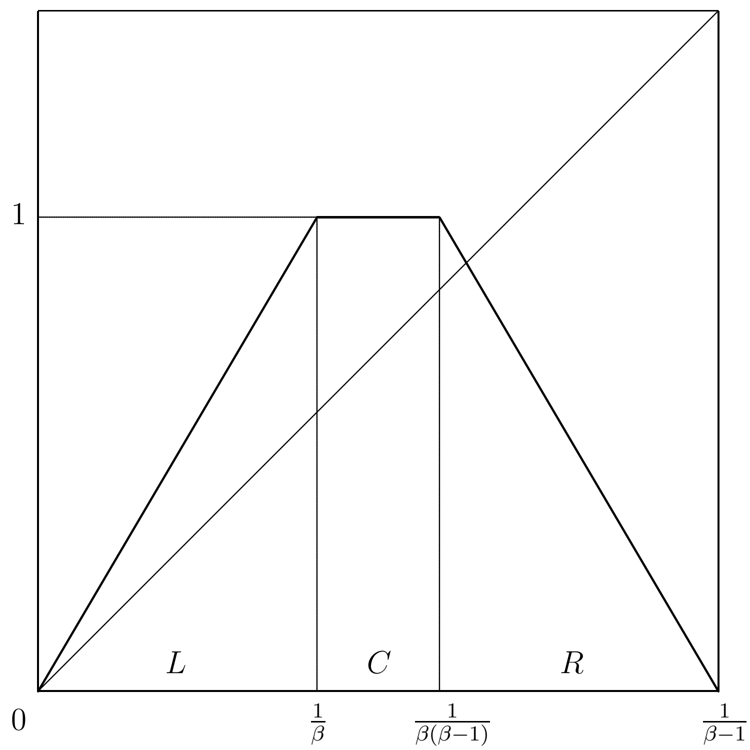

FIGURE 7. The trapezoidal map S_β for $\beta = 1.7$

Let $\prec_u$ denote the *unimodal order* on the itineraries of T_β, i.e., $L \prec_u C \prec_u R$ and $\varepsilon \prec_u \varepsilon'$ if $\varepsilon_i \equiv \varepsilon'_i$, $1 \le i \le k$ and either $\varepsilon_{k+1} \prec_u \varepsilon'_{k+1}$ with $\#\{i \in [1,k] : \varepsilon_i = R\}$ even or $\varepsilon_{k+1} \succ_u \varepsilon'_{k+1}$ with $\#\{i \in [1,k] : \varepsilon_i = R\}$ odd.

We have for $\varepsilon, \varepsilon' \in \Sigma$,

$$\varepsilon \prec \varepsilon' \iff h(\varepsilon) \prec_u h(\varepsilon').$$

The map h helps to prove our version of the Sharkovskiĭ theorem via the classical one.

2.1. **Finite number of beta-expansions.** Put

$$\mathcal{B}_m = \{\beta \in (G,2) : \exists x \in [0, 1/(\beta-1)] \text{which has exactly } m \text{ expansions in base } \beta\}.$$

Lemma 7. *We have $\mathcal{B}_m \subset \mathcal{B}_2$ for $m \ge 3$ and $m \in \mathbb{N}$.*

Hence if $\beta \notin \mathcal{B}_2$, then we have the following *dichotomy*: *either a number $x \in J_\beta$ has a unique β-expansion or infinitely many of them.*

Theorem 8 (N. Sidorov, 2009). *The smallest element of $\mathcal{B}_2$ is $\widetilde{\beta}_2$, the appropriate root of $x^4 = 2x^2 + x + 1$, with the numerical value $\widetilde{\beta}_2 \approx 1.71064$. Furthermore, $\mathcal{B}_2 \cap (\widetilde{\beta}_2, \beta_4) = \emptyset$.*

Here, as above, $\beta_4 \approx 1.75488$ is the appropriate root of $x^3 = 2x^2 - x + 1$.

Theorem 9. *For $\beta \in (G, \beta')$ the* **strong** *dichotomy holds provided β is* transcendental.

(*Strong dichotomy* means that any x has either a unique β-expansion or a continuum of them.)

So, we know that $\mathcal{B}_2 \cap (G, \beta')$ is countable (*lower order*).

Theorem 10 (middle order). *The set $\mathcal{B}_2 \cap (\beta', \beta' + \delta)$ has the cardinality of the continuum for any $\delta > 0$.*

Theorem 11 (top order). *Let, as above, β_3 denote the root of $x^3 = x^2 + x + 1$, $T \approx 1.83929$. Then $[\beta_3, 2) \subset \mathcal{B}_2$, i.e., there always x which has exactly two β-expansions provided $\beta \geq \beta_3$.*

A similar result holds for $\mathcal{B}_m$ for any $m \geq 3$.

3. Topology of sums in nonnegative powers of $\beta > 1$

Let $1 < \beta < 2$ be our parameter. Put

$$\Lambda_n(\beta) = \left\{ \sum_{k=0}^{n} a_k \beta^k \mid a_k \in \{-1, 0, 1\} \right\}$$

and

$$\Lambda(\beta) = \bigcup_{n \geq 1} \Lambda_n(\beta).$$

Trivial properties of $\Lambda(\beta)$:

- countable;
- unbounded;
- symmetric about 0;

Question: what is the *topology* of $\Lambda(\beta)$? Is it dense? discrete? neither?

Theorem 12 (Garsia, 1962 [13]). *Let β be a Pisot number, i.e, an algebraic integer whose other conjugates are less than 1 in modulus. Then $\Lambda(\beta)$ is uniformly discrete.*

Proof. Without loss of generality we may assume $x, y \in \Lambda_n(\beta)$ and $x \neq y$. Then $x - y = \sum_0^n \varepsilon_k \beta^k$ with $\varepsilon_k \in \{-2, -1, 0, 1, 2\}$. Put

$$P(t) = \sum_0^n \varepsilon_k t^k.$$

Let $\beta_1 = \beta, \beta_2, \dots, \beta_d$ be the conjugates of β. Since $P(\beta) \neq 0$, we have $P(\beta_j) \neq 0$ for all j. Hence $\prod_1^d P(\beta_j) \neq 0$. As this product is an integer (exercise!), we have

$$\left| \prod_1^d P(\beta_j) \right| \geq 1.$$

Consequently,

$$|P(\beta)| \geq \frac{1}{\left|\prod_{j\geq 2} P(\beta_j)\right|}.$$

Since $|\beta_j| < 1$ for all $j \geq 2$ (Pisot!), we have

$$\left|\sum_{i=0}^{n} \varepsilon_i \beta_j^i\right| = O(1),$$

whence $|P(\beta)| \geq$ const. □

Theorem 13 (folklore)**.** *If β is transcendental, then 0 is a limit point of $\Lambda(\beta)$.*

Proof. Put

$$D_n(\beta) = \left\{\sum_{k=0}^{n} a_k \beta^k \mid a_k \in \{0,1\}\right\}.$$

Since β is transcendental, $z_n(\beta) := \#D_n(\beta) = 2^{n+1}$. On the other hand, $\max D_n(\beta) = O(\beta^n) \ll 2^n$.

By the pigeonhole principle, there exist $x, y \in D_n(\beta)$ such that

$$|x - y| \leq \text{const} \cdot \left(\frac{\beta}{2}\right)^n = o(1).$$

Since $x - y \in \Lambda_n(\beta)$, we are done. □

Theorem 14 (Drobot, 1973 [6])**.** *If 0 is a limit point of $\Lambda(\beta)$, then $\Lambda(\beta)$ is dense in $\mathbb{R}$.*

Thus, if β is not of *height 1* (i.e., is not a root of $-1, 0, 1$ polynomial), then $\Lambda(\beta)$ is dense. (For example, $\beta = \sqrt{2}$.)

Conjecture. If β is not Pisot, then $z_n(\beta) \gg \beta^n$ and consequently, $\Lambda(\beta)$ is dense.

Definition 15. We say that an algebraic $\beta > 1$ is a *Perron number* if $|\alpha| < \beta$ for any conjugate α of β.

Theorem 16 (Sidorov and Solomyak, 2009 [21])**.** *If β is not Perron, then $\Lambda(\beta)$ is dense in $\mathbb{R}$.*

Proof. Here is a crude idea of our proof: assume there exists α which is a conjugate of β such that $\beta < |\alpha|$. It is easy to see that $z_n(\beta) = z_n(\alpha)$ (since there is a natural bijection between the sets $D_n(\beta)$ and $D_n(\alpha)$). Then we show that $z_n(\alpha) \geq \text{const} \cdot |\alpha|^n$ (this is the key point of our proof), whence $z_n(\beta) \gg \beta^n$, and we apply the pigeonhole principle. □

Let $D(\beta)$ denote the set of all finite 0-1 sums in nonnegative powers of β, i.e., $D(\beta) = \bigcup_{n\geq 1} D_n(\beta)$. Since for any $E > 0$ we have that $[0, E] \cap D(\beta)$ is finite, $D(\beta)$ is discrete.

Write

$$D(\beta) = \{y_0(\beta) < y_1(\beta) < \dots\}.$$

Put

$$\ell(\beta) = \liminf_n (y_{n+1} - y_n) \tag{3.1}$$

and

$$L(\beta) = \limsup_n (y_{n+1} - y_n).$$

It is obvious that $\ell(\beta) = 0$ if and only if 0 is a limit point of $\Lambda(\beta)$. Hence $\ell(\beta) = 0 \iff \Lambda(\beta)$ is dense in $\mathbb{R}$.

Theorem 17 (Erdős and Komornik, 1998 [10])**.** *For any $\beta < 2^{1/4}$ we have $L(\beta) = 0$.*

It is also known that $L(\sqrt{2}) = 0$ and $L(\beta) = \beta$ for any $\beta \geq \frac{1+\sqrt{5}}{2}$ (see Problem Sheet 2). No $\beta \in \left(\sqrt{2}, \frac{1+\sqrt{5}}{2}\right)$ with $L(\beta) = 0$ is known.

4. Bernoulli convolutions

Let $\beta > 1$ and define the *Bernoulli convolution* ξ_β as follows. Let $b_n(\beta)$ be the two-point distribution such that $b_n(-\beta^{-n}) = b_n(\beta^{-n}) = 1/2$. Now

$$\xi_\beta = b_1(\beta) * b_2(\beta) * \dots,$$

an infinite convolution. Note that $b_1(\beta) * b_2(\beta) * \cdots * b_n(\beta)$ is supported by the finite set $\left\{\sum_{k=1}^n \varepsilon_k \beta^{-k} : \varepsilon_k \in \{-1, 1\}\right\}$ and each point has the measure 2^{-n}. (Some of them may coincide is β is algebraic.) Hence for any Borel set $E \subset \mathbb{R}$,

$$\xi_\beta(E) = \mathbb{P}\left\{(a_1, a_2, \dots) \in \{-1, 1\}^{\mathbb{N}} : \sum_{k=1}^{\infty} a_k \beta^{-k} \in E\right\},$$

where $\mathbb{P}$ is the product measure on $\{-1, 1\}^{\mathbb{N}}$ with $\mathbb{P}(a_1 = -1) = \mathbb{P}(a_1 = 1) = 1/2$.

The reason people have got interested in Bernoulli convolutions in the 1930s (see [23] for a comprehensive survey) is their especially nice Fourier transform:

$$\begin{aligned}\widehat{\xi_\beta}(x) &= \prod_{n=1}^{\infty} \frac{1}{2}\left(e^{-i\beta^{-n}x} + e^{i\beta^{-n}x}\right) \\ &= \prod_{n=1}^{\infty} \cos(\beta^{-n}x).\end{aligned}$$

We also define the measure ν_β in a similar way (replacing -1 with 0):

$$\nu_\beta(E) = \mathbb{P}\left\{(a_1, a_2, \dots) \in \{0, 1\}^{\mathbb{N}} : \sum_{k=1}^{\infty} a_k \beta^{-k} \in E\right\}.$$

In other words, ν_β "measures" how many β-expansions fall into a given set. It is easy to see that ν_β is a scaled copy of ξ_β (exercise!), so their important properties should be the same.

Recall that a measure ν is called *absolutely continuous* (with respect to the Lebesgue measure $\mathcal{L}$) if $\mathcal{L}(E) = 0$ implies $\nu(E) = 0$. In this case there exists an integrable function h (the Radon-Nikodym density) such that $\nu(E) = \int_E h(x)\,dx$.

A measure ν is called *singular* if there exists a Borel set F such that $\nu(F) = 0$ and $\mathcal{L}(F) = 1$. (Here $\mathcal{L}$ is a probability measure.)

Theorem 18 (Jessen-Wintner, 1935)**.** *For any $\beta > 1$ the measure ν_β is either absolutely continuous or singular.*

This result is often referred to as the *Law of Pure Types.*

Note that if $\beta = 2$, then ν_β is none other than the Lebesgue measure. If $\beta > 2$, then ν_β "sits" on a Cantor set of zero Lebesgue measure (exercise!) and hence is singular. But what happens if $\beta \in (1, 2)$?

Definition 19. An algebraic integer $\beta > 1$ is called a *Pisot number* (or a Pisot-Vijayaraghavan (PV) number) if all its other Galois conjugates are less than 1 in modulus.

The set of Pisot numbers is known to be closed (sic!). The smallest Pisot number is the real root of $x^3 - x - 1$. The smallest limit point of the set of Pisot numbers is the golden ratio. The main property of a Pisot number β is that there exists a sequence of positive integers z_N such that

$$\beta^N = z_N + O(\gamma^N), \quad N \to +\infty \tag{4.1}$$

for some $\gamma \in (0, 1)$.

Recall the Riemann-Lebesgue Lemma (or Theorem in some textbooks): for any f in $L^1(\mathbb{R})$ we have $\widehat{f}(x) \to 0$ as $x \to \pm\infty$. Consequently, for any absolutely continuous measure ν we have $\widehat{\nu}(x) \to 0$ as $x \to \pm\infty$.

Theorem 20 (Erdős, 1939 [7])**.** *For any Pisot $\beta \in (1, 2)$ the Bernoulli convolution ξ_β is singular.*

Proof. We will show that $\widehat{\xi_\beta}(x) \not\to 0$ as $x \to +\infty$, which will imply that ξ_β cannot be absolutely continuous. Therefore, by the Law of Pure Types, it must be singular.

Put $x_N = 2\pi\beta^N$. We have

$$\begin{aligned}\widehat{\xi_\beta}(x_N) &= \prod_{n=1}^{\infty} \cos(2\pi\beta^{N-n}x) \\ &= \cos(2\pi\beta^N) \cdot \cos(2\pi\beta^{N-1}) \cdots \cos(2\pi\beta) \cdot \widehat{\xi_\beta}(2\pi).\end{aligned}$$

Since β is irrational, $\widehat{\xi_\beta}(2\pi) \neq 0$ (check it!). In view of (4.1), $\cos(2\pi\beta^k) = \cos(2\pi\beta^k - 2\pi z_k) = 1 - O(\gamma^k)$. Hence

$$|\cos(2\pi\beta^N) \cdot \cos(2\pi\beta^{N-1}) \cdots \cos(2\pi\beta)| \geq \text{const},$$

whence

$$|\widehat{\nu}_\beta(x_N)| \geq \text{const}'.$$

□

There exists an alternative proof [19] in which we construct a measure $\widetilde{\nu}_\beta$ which is equivalent to ν_β such that the greedy β-transformation preserves it, and it is ergodic.

Theorem 21 (B. Solomyak, 1995 [22])**.** *For Lebesgue-a.e. $\beta \in (1,2)$ the Bernoulli convolution ξ_β is absolutely continuous.*

There is only one explicit family of β for which it is known that ξ_β is absolutely continuous.

Definition 22. An algebraic integer $\beta > 1$ is called a *Garsia number* if all its Galois conjugates are greater than 1 in modulus, and the constant term of its minimal polynomial is ± 2.

Such is $\sqrt{2}$ or the appropriate root of $x^4 - x - 2$, say.

Theorem 23 (Garsia, 1962 [13])**.** *For any Garsia β the Bernoulli convolution ξ_β is absolutely continuous with a bounded density.*

5. Multidimensional β-expansions

Let, as above, $\beta > 1$ be our parameter. Consider a pair of maps (similitudes) in the real line:

$$\begin{aligned} f_0(x) &= x/\beta, \\ f_1(x) &= x/\beta + 1. \end{aligned}$$

They constitute an *iterated function system* (IFS). That is, choose 0 as a starting point, and for any sequence $(\varepsilon_1, \varepsilon_2, \dots)$ of 0s and 1s:

$$x = \lim_{N \to +\infty} f_{\varepsilon_1} \dots f_{\varepsilon_N}(0).$$

The set of all x's that are representable in such a form, is called the *invariant set* I_β of the IFS.

Unlike a general IFS (see, e.g., [11]), in our model this expression can be given in a very simple form:

$$\begin{aligned} f_{\varepsilon_1} \dots f_{\varepsilon_N}(0) &= \beta^{-1}\varepsilon_1 + \beta^{-1}(\varepsilon_2 \\ &+ \beta^{-1}(\varepsilon_3 + \dots + \beta^{-1}\varepsilon_N)\dots)) \\ &= \sum_{n=1}^{N} \varepsilon_n \beta^{-n}, \end{aligned}$$

whence

$$x = \lim_N \sum_{n=1}^{N} \varepsilon_k \beta^{-n} = \sum_{n=1}^{\infty} \varepsilon_n \beta^{-n}.$$

We see that the invariant set is none other than the set of β-expansions.

Let $\boldsymbol{p}_0, \dots, \boldsymbol{p}_k$ now be points in $\mathbb{R}^d$. Consider the IFS – a general collection of similitudes:

$$f_i(\boldsymbol{x}) = \beta^{-1}\boldsymbol{x} + (1 - \beta^{-1})\boldsymbol{p}_i. \tag{5.1}$$

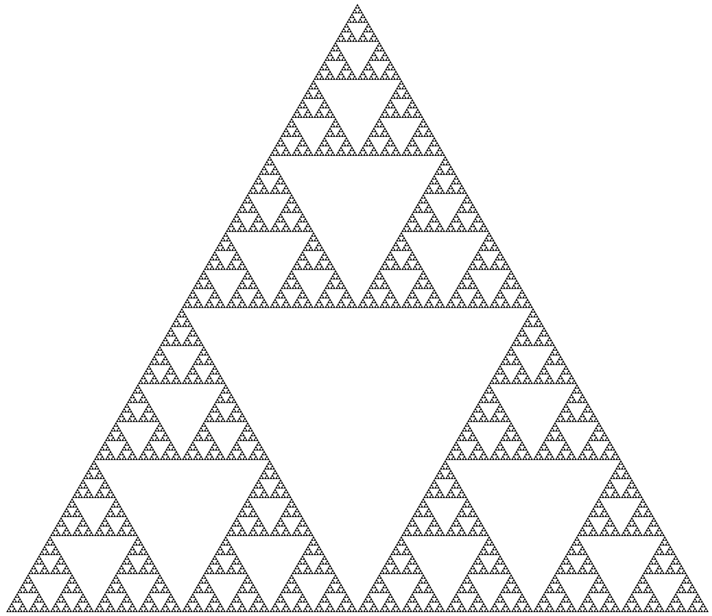

FIGURE 8. The Sierpiński Gasket

Then any point $\boldsymbol{x}$ in the invariant set has a representation in the form

$$\boldsymbol{x} = (\beta - 1)\sum_{n=1}^{\infty} \beta^{-n}\boldsymbol{a}_n,$$

where $\boldsymbol{a}_n$ is one of the vertices $\boldsymbol{p}_i$.

Unlike the one-dimensional case, the invariant set J_β (which lies in the convex hull of the set $\{\boldsymbol{p}_0, \dots, \boldsymbol{p}_k\}$) may have a complicated structure.

Let $\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2$ be the vertices of a triangle Δ in $\mathbb{R}^2$ (equilateral, say—this does not matter!). Note first that if $\beta \le 3/2$, then $J_\beta = \Delta$. If $\beta \in (3/2, 2)$, then we have both holes and overlaps.

The most famous case is $\beta = 2$ – see Figure 8. Its Hausdorff dimension is known to be equal to $\log 3/\log 2$.

Assume now $\beta \in (3/2, 2)$. Let first $\beta = \frac{1+\sqrt{5}}{2}$. We get the following nice fractal – see Figure 9.

Theorem 24 (D. Broomhead, J. Montaldi and N. Sidorov, 2003 [3])**.** *The invariant set J_β is **totally self-similar**, i.e.,*

$$f_{\varepsilon_0} \dots f_{\varepsilon_{n-1}}(J_\beta) = f_{\varepsilon_0} \dots f_{\varepsilon_{n-1}}(\Delta) \cap J_\beta$$

for any $\varepsilon_0, \dots, \varepsilon_{n-1}$.

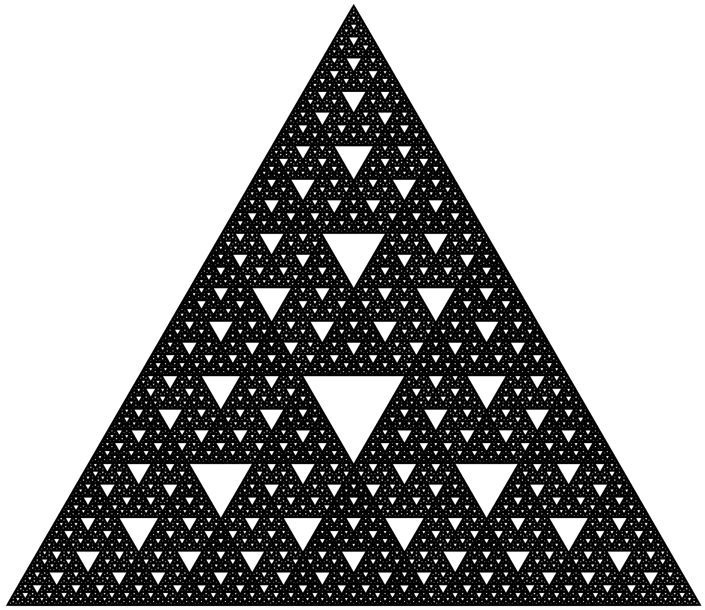

FIGURE 9. The Golden Gasket

Theorem 25 (D. Broomhead, J. Montaldi and N. Sidorov, 2003 [3])**.**

$$\dim_H(J_\beta) = -\frac{\log\tau}{\log\beta} = 1.93063\ldots,$$

where where $\tau \approx 0.39493$ *is a root of the polynomial* $3z^3 - 3z + 1$*, namely,*

$$\tau = \frac{2}{\sqrt{3}}\cos(7\pi/18).$$

Theorem 26 (D. Broomhead, J. Montaldi and N. Sidorov, 2003 [3])**.** *If the invariant set* J_β *is totally self-similar for some* $\beta \in (3/2, 2)$*, then* β *satisfies*

$$\beta^m = \beta^{m-1} + \beta^{m-2} + \cdots + \beta + 1$$

for some $m \geq 2$ *(***multinacci numbers***).*

Here is a sketch of the proof of the key Theorem 24 (for an arbitrary multinacci β). Let x, y, z be the distances to the sides of Δ so that $x+y+z = 1$. These are called *barycentric coordinates*.

Then the f_i are linear maps in barycentric coordinates, and one can easily check that

$$f_0 = \begin{pmatrix} 1 & 1-\lambda & 1-\lambda \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

$$f_1 = \begin{pmatrix} \lambda & 0 & 0 \\ 1-\lambda & 1 & 1-\lambda \\ 0 & 0 & \lambda \end{pmatrix},$$

$$f_2 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 1-\lambda & 1-\lambda & 1 \end{pmatrix},$$

where $\lambda = \beta^{-1}$. Moreover,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lim_{N\to+\infty} f_{\varepsilon_0} \dots f_{\varepsilon_N}(\mathbf{0}) = \begin{pmatrix} (\beta-1)\sum_{k=1}^\infty a_k\beta^{-k} \\ (\beta-1)\sum_{k=1}^\infty b_k\beta^{-k} \\ (\beta-1)\sum_{k=1}^\infty c_k\beta^{-k} \end{pmatrix},$$

where $a_k, b_k, c_k \in \{0,1\}$ and $a_k + b_k + c_k = 1$. (In fact, $a_k = \chi_{\{\varepsilon_k=0\}}, \chi_{\{\varepsilon_k=1\}}, \chi_{\{\varepsilon_k=2\}}$.) Let $\Delta_0 = \Delta$, and

$$\Delta_n = \bigcup_{i=0}^{2} f_i(\Delta_{n-1}), \quad n \ge 1.$$

The *central hole* $H_0 := \Delta \setminus \Delta_1$. Then each hole is a subset of an image of H_0.

The key to the proof is the fact that for the multinacci β any image of the central hole is a hole. This is easily equivalent to the total self-similarity of J_β.

It suffices to show that $H_n := f_{\varepsilon_0} \dots f_{\varepsilon_{n-1}}(H_0)$ has an empty intersection with Δ_{n+1}. This is equivalent to the fact that the system

$$\beta^{-n-1} + \sum_1^{n-1} a_k\beta^{-k} > \sum_1^n a'_k\beta^{-k},$$

$$\beta^{-n-1} + \sum_1^{n-1} b_k\beta^{-k} > \sum_1^n b'_k\beta^{-k},$$

$$\beta^{-n-1} + \sum_1^{n-1} c_k\beta^{-k} > \sum_1^n c'_k\beta^{-k}$$

does not have a solution. This in turn follows from

Theorem 27 (P. Erdős, I. Joó, M. Joó, 1992 [8]). *Let $\ell(\beta)$ be given by (3.1). Then $\ell(\beta) = \beta^{-1}$ if β is a multinacci number.*

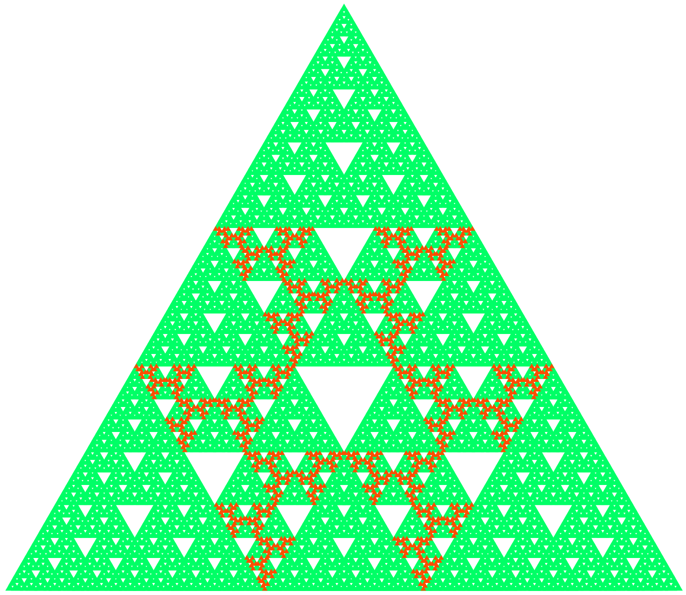

FIGURE 10. The set of uniqueness superimposed on the golden gasket

In other words, β^{-1} is the exact separation constant in the Garsia separation lemma (Theorem 12) if β is multinacci. See Figure 10 for the set of uniqueness for the golden gasket.

The main problem remaining is to determine for which β the attractor J_β has positive two-dimensional Lebesgue measure and for which zero Lebesgue measure.

Theorem 28. [3] *For β sufficiently close to $3/2$ the measure is positive and, moreover, the interior of J_β is nonempty. For $\beta > \sqrt{3}$ the measure of J_β is zero.*

The numerics suggests the following

Conjecture. (1) For each $\beta \in \left(\frac{3}{2}, \frac{1+\sqrt{5}}{2}\right)$ the attractor J_β has a nonempty interior – see Figure 11.
(2) For each $\beta \in \left(\frac{1+\sqrt{5}}{2}, \sqrt{3}\right)$ it has an empty interior – see Figure 12.

Return to the general setting (5.1). There exists an analogue of Theorem 1:

Theorem 29 (Sidorov, 2007 [20])**.** *For each $\boldsymbol{p}_0, \dots, \boldsymbol{p}_{m-1}$ there exists $\beta_0 > 1$ such that for any $\beta > \beta_0$,*

1. *There are no holes in J_β.*
2. *Each point $\boldsymbol{x}$ in the convex hull of $\{\boldsymbol{p}_0, \dots, \boldsymbol{p}_{m-1}\}$ except when $\boldsymbol{x}$ is $\boldsymbol{p}_i$, has $2^{\aleph_0}$ distinct addresses.*

Thus, β_0 in this theorem is a direct analogue of the golden ratio in the one-dimensional setting. To determine the sharp value of β_0 for a given collection $\{\boldsymbol{p}_0, \dots, \boldsymbol{p}_{m-1}\}$ is an interesting problem.

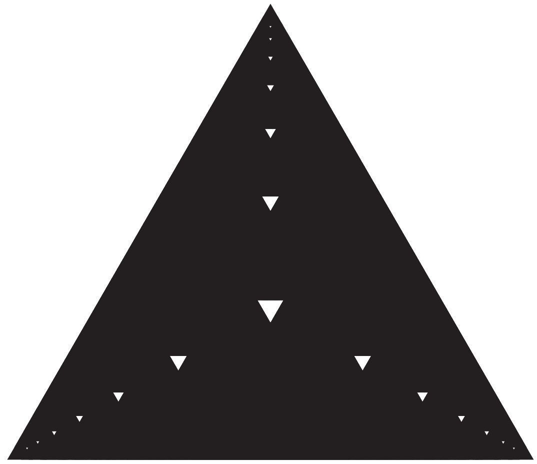

FIGURE 11. The invariant set J_β for $\beta = 1.54$

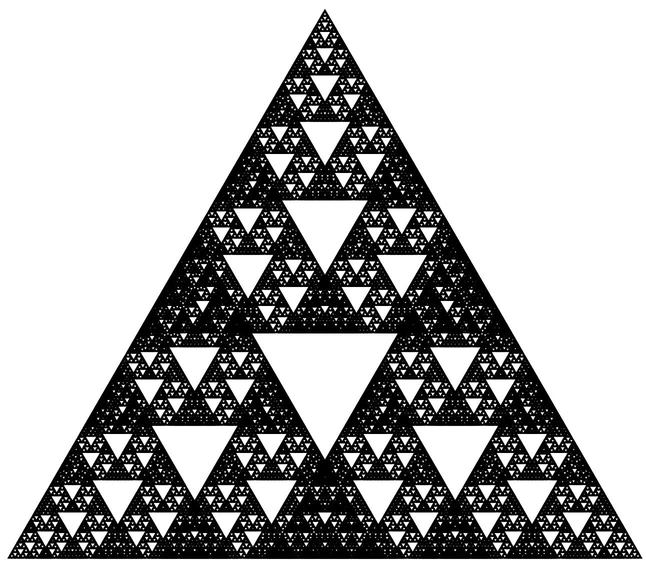

FIGURE 12. The invariant set J_β for $\beta = 1.69$

There also exists a multidimensional generalization of Theorem 3:

Theorem 30. [20] *Assume that the attractor J_β has no holes plus some technical condition. Then Lebesgue-a.e. $\boldsymbol{x}$ in the convex hull of the $\boldsymbol{p}_i$ has a continuum of distinct β-expansions, and the exceptional set has Hausdorff dimension strictly less than d, the dimension of the convex hull of the $\boldsymbol{p}_i$.*

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