1. Introduction into $\beta$-expansions

Representations of real numbers in non-integer bases were introduced by Rényi [17] and first studied by Rényi and by Parry [16].

Let first $\beta$ be an integer greater than 1. Then any number $x \in [0, 1)$ can be represented in the form

$$x = \sum_{n=1}^{\infty} a_n \beta^{-n}, \quad a_n \in \{0, 1, \ldots, \beta\}.$$ 

This representation is unique, except for a countable set of $x$. The corresponding map here is $\tau_\beta : [0, 1) \to [0, 1)$ defined by the formula

$$\tau_\beta(x) = \beta x \mod 1.$$ 

This map acts as the shift on the expansions, i.e., $a_n(\tau_\beta x) = a_{n+1}(x)$. The properties of this map are well known; in particular, it preserves the Lebesgue measure on the interval, and the corresponding dynamical system has various nice properties. See Figure 1 for the case $\beta = 2$.

Assume now $\beta > 1$ to be non-integer. We call any representation of the form

$$x = \sum_{n=1}^{\infty} a_n \beta^{-n}, \quad a_n \in \{0, 1, \ldots, \lfloor \beta \rfloor - 1\}.$$ 

a $\beta$-expansion of $x$. (Here $[t]$ denotes the integer part of $t$.) For instance, for $\beta \in (1, 2)$ – which is going to be our main example – the set of “digits” is $\{0, 1\}$, i.e., like the one for the binary expansions. It is easy to show “by hand” that any $x \in \left[0, \frac{\lfloor \beta \rfloor}{\beta - 1}\right]$ has at least one $\beta$-expansion.

We will do it in a way similar to the standard doubling map. Let us assume for simplicity that $1 < \beta < 2$ and introduce the following multivalued map:

$$T_\beta(x) = \begin{cases} 
\beta x, & x \in \left[0, \frac{1}{\beta}\right] \\
\beta x \text{ or } \beta x - 1, & x \in \left(\frac{1}{\beta}, \frac{1}{\beta (\beta - 1)}\right) \\
\beta x - 1, & x \in \left[\frac{1}{\beta (\beta - 1)}, \frac{1}{\beta - 1}\right]
\end{cases}$$

(see Figure 2).

Date: July 28, 2010.
We see that if \( x \in [0, \frac{1}{\beta}) \) or \( x \in \left( \frac{1}{\beta(\beta-1)}, \frac{1}{\beta-1} \right) \), then \( T_\beta(x) \) is uniquely defined. However, whenever \( x \) lies in the **switch region** \( \left[ \frac{1}{\beta}, \frac{1}{\beta(\beta-1)} \right] \), we have a choice between 0 and 1.

Figure 3 depicts a branching pattern that occurs for the multivalued map \( T_\beta \). We will see that typically it is indeed a binary tree.

If we always choose 1 (or, in the general case, the largest possible “digit”), such an expansion is called **greedy**. The map \( T_\beta \) becomes the \( \beta \)-transformation \( \tau_\beta x = \beta x \mod 1 \) (restricted to \([0, 1)\)) – see Figure 4.

Although \( \tau_\beta \) does not preserve the Lebesgue measure, there exists a bounded positive density function \( h_\beta \) such that the absolutely continuous measure \( \mu_\beta \) given by \( h_\beta \) is \( \tau_\beta \)-invariant (see [16]). The dynamical system \(([0, 1), \mu_\beta, \tau_\beta)\) is well studied, and its properties are similar to the ones of the doubling map.

**Theorem 1.** ([9]) If \( \beta < \frac{1+\sqrt{5}}{2} \), then any \( x \in (0, 1/(\beta - 1)) \) has a continuum of distinct \( \beta \)-expansions.

**Proof.** One can check (exercise!) that if \( x < 1/\beta \), then it is impossible that \( T_\beta(x) > 1/(\beta(\beta - 1)) \) – see Figure 5. Hence eventually the trajectory of any point bifurcates, and the procedure repeats for each of the images, ad infinitum. \( \square \)
A quantitative version of this result has been recently proven by Feng and the author. Put

\[ \mathcal{N}_n(x; \beta) = \# \left\{ (a_1, \ldots, a_n) \in \{0, 1\}^n \mid \exists (a_{n+1}, a_{n+2}, \ldots) : x = \sum_{k=1}^{\infty} a_k \beta^{-k} \right\}. \]
Theorem 2. ([12]) Let $\beta$ be an arbitrary number in $(1, \frac{1+\sqrt{5}}{2})$. Then there exists $c = c(\beta) > 0$ such that

$$\liminf_{n \to \infty} \frac{\log N_n(x; \beta)}{n} \geq c$$

for any $x \in (0, \frac{1}{\beta - 1})$.

What about when $\beta$ is greater than the golden ratio? In this case one can show (exercise!) that there exists a point $x = x(\beta) < 1/\beta$ such that $T_\beta(x) > 1/(\beta(\beta - 1))$, and $T_\beta(x) = x$ (a 2-cycle) – see Figure 6.

Hence the $\beta$-expansion of such a point is necessarily 010101… We will discuss unique $\beta$-expansions in detail in the next section.

Thus, it is not true that every internal point has a continuum of $\beta$-expansions if $\beta$ is between the golden ratio and 2. However, a weaker result is still valid:

Theorem 3. (Sidorov [18, 19])

1. Almost every point $x \in (0, 1/(\beta - 1))$ has a continuum of $\beta$-expansions.
2. Furthermore, the set of exceptions has Hausdorff dimension strictly less than 1.

Proof. We will prove the first part. Our first goal is to show that a.e. $x \in (0, 1)$ has at least two different $\beta$-expansions. We may assume that $\beta \geq \frac{1+\sqrt{5}}{2}$. 

\[1\]
Figure 5. The $\beta$-transformation $T_\beta$ for $\beta = 1.25$

Figure 6. The 2-cycle
Since $\beta$ belongs to $[(1 + \sqrt{5})/2, 2)$, there exists $m = m(\beta) \geq 2$ such that

$$1 + \beta^{-m+1} < \frac{1}{\beta - 1};$$

specifically, we can take

$$m = \left\lfloor \log_\beta \frac{\beta - 1}{2 - \beta} \right\rfloor + 1 \geq 2$$

(for $\beta = (1 + \sqrt{5})/2$ we have $\beta - 1 = \beta^{-1}, 2 - \beta = \beta^{-2}$, whence $\log_\beta \frac{\beta - 1}{2 - \beta} = 1$).

So, we consider $\varepsilon$ in $(0, 1)$, and assume that its greedy expansion is of the form

$$(\varepsilon_1, \ldots, \varepsilon_n, 1, 0, \ldots, 0, \varepsilon_{n+m+1}, \ldots).$$

We can construct a different $\beta$-expansion for $\varepsilon$. Namely, if $x' = \sum_{j=1}^{n} \varepsilon_j \beta^{-j}$, then

$$x - x' = \beta^{-n-1} + \sum_{j=n+m+1}^{\infty} \varepsilon_j \beta^{-j} \in [\beta^{-n-1}, \beta^{-n-1} + \beta^{-n-m}],$$

because $\sum_{n+m+1}^{\infty} \varepsilon_j \beta^{-j} \leq \beta^{-n-m}$ (a property of the greedy expansions). On the other hand, we infer from (1.1) that

$$\beta^{-n-1} + \beta^{-n-m} < \beta^{-n-2} + \beta^{-n-3} + \ldots = \frac{\beta^{-n-1}}{\beta - 1},$$

whence

$$x - x' < \beta^{-n-2} + \beta^{-n-3} + \ldots$$

as well. This means that if we put $\varepsilon_{n+1}' = 0$, it is possible to find $(\varepsilon_{n+2}', \varepsilon_{n+3}', \ldots)$ in $\Sigma$ such that $x = \sum_{j=1}^{\infty} \varepsilon_j' \beta^{-j}$. By our construction, $\varepsilon_{n+1}' \neq \varepsilon_{n+1}$.

Thus, the set $U_\beta$ – all $\varepsilon$ which have a unique $\beta$-expansion – has measure zero. Now, if for some $\varepsilon$ its tree of $\beta$-expansions (see Figure 3) is not the full binary tree, it means that one of the branches “flatlines”. This implies that for one of $\beta$-expansions of $\varepsilon$, say, for $(\varepsilon_1, \varepsilon_2, \ldots)$, there exists $k$ such that $(\varepsilon_k, \varepsilon_{k+1}, \ldots)$ is a unique expansion (since it does not bifurcates any further).

Since any shift of a $\beta$-expansion is either $\beta x$ or $\beta x - 1$, we infer that $x$ belongs to a scaled copy of $U_\beta$. Any such copy has zero measure and there is only a countable set of them for $x$ to lie in. Hence the set of $x$ whose branching is not full is a zero measure set. In particular, a.e. $x$ has a continuum of $\beta$-expansions. \hfill \Box

Finally, we would like to mention random $\beta$-expansions. Again, we assume for simplicity that $1 < \beta < 2$. Put $\Omega = \{0, 1\}^\mathbb{N}$, and we regard $0$ as “tails” and $1$ as “heads”. We introduce the random $\beta$-transformation $K_\beta : [0, \frac{1}{\beta - 1}] \times \Omega \to [0, \frac{1}{\beta - 1}] \times \Omega$ as follows:

$$K_\beta(x, \omega) = \begin{cases} (\beta x, \omega), & x \in [0, \frac{1}{\beta}] \\ (\beta x - \omega_1, \sigma(\omega)), & x \in \left[\frac{1}{\beta}, \frac{1}{m(\beta - 1)}\right] \\ (\beta x - 1, \omega), & x \in \left(\frac{1}{m(\beta - 1)}, \frac{1}{\beta - 1}\right) \end{cases}$$
Here $\sigma : \Omega \to \Omega$ is the one-sided shift, i.e., $\sigma(\omega_1, \omega_2, \omega_3, \ldots) = (\omega_2, \omega_3, \ldots)$. In other words, if we are outside the switch region, we just apply $\beta x$ or $\beta x - 1$ respectively and do not touch the "coin". If we are in the switch region, we flip a coin (= check $\omega_1$) and apply the corresponding map, after which we shift $\omega$ for the next flip, whenever we’ll need it.

It has been shown in [4] that there exists a unique probability measure $m_\beta$ on $[0, \frac{1}{\beta-1}]$ such that $m_\beta$ is equivalent to the Lebesgue measure and $m_\beta \otimes P$ is invariant and ergodic under $K_\beta$, where $P = \prod_{1}^{\infty} \{ \frac{1}{2}, \frac{1}{2} \}$.

2. Unique $\beta$-expansions and their dynamics

Let, as above, $U_\beta$ denote the set of $x \in (0, 1/(\beta - 1))$ which have a unique $\beta$-expansion. Put $G = \frac{1 + \sqrt{5}}{2}$.

**Theorem 4** (Glendinning-Sidorov, 2001 [14]). We have the following dichotomy:

- The set $U_\beta$ is infinite countable if $\beta \in (G, \beta')$, and each unique expansion is eventually periodic.
- If $\beta \in (\beta', 2)$, then $U_\beta$ has the cardinality of the continuum and a positive Hausdorff dimension.

Here $\beta'$ is the Komornik-Loreti constant which is defined as follows: denote by

$$(m_k)_{k=0}^{\infty} = 0110 \ 1001 \ 0110 \ 1001 \ldots$$

the Thue-Morse sequence, i.e., the fixed point of the substitution $0 \to 01, 1 \to 10$.

The **Komornik-Loreti constant** $\beta' \approx 1.78723$ is defined as the unique solution of the equation

$$\sum_{k=1}^{\infty} m_k x^{-k} = 1.$$

This constant proves to be the smallest $\beta$ such that $1 \in U_\beta$. Allouche and Cosnard [2] have proved that $\beta'$ is transcendental.

The topology of $U_\beta$ can be complicated, depending on $\beta$. For some $\beta$ it is a Cantor set, for some it isn’t. For more detail see [15].

The set $U_\beta$ is invariant under $T_\beta$ (why?), hence we can consider $F_\beta = T_\beta|_{U_\beta}$. Recall the Sharkovskii order on $\mathbb{N}$:

$$3 \triangleright 5 \triangleright 7 \triangleright \cdots \triangleright 2m+1 \triangleright \cdots \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright \cdots \triangleright 2 \cdot (2m+1) \triangleright \cdots \triangleright 4 \cdot 3 \triangleright 4 \cdot 5 \triangleright 4 \cdot 7 \triangleright \cdots \triangleright 4 \cdot (2m+1) \triangleright \cdots \triangleright 2^n \cdot 3 \triangleright 2^n \cdot 5 \triangleright 2^n \cdot 7 \triangleright \cdots \triangleright 2^n \cdot (2m+1) \triangleright \cdots \triangleright \cdots \triangleright 8 \triangleright 4 \triangleright 2 \triangleright 1,$$

where the relation $a \triangleright b$ indicates that $a$ comes before $b$ in the ordering.
Theorem 5 ((Sharkovskii’s Theorem), see [5]). Let $f$ be a continuous automorphism of a compact interval $I$. If $k \succ l$ in Sharkovskii’s ordering and if $f$ has a point of smallest period $k$, then $f$ also has a point of smallest period $l$.

Now we are ready to state the main theorem of the this section. Put

$$ U_n = \{ \beta \in (1, 2) : F_\beta \text{ has an } n\text{-cycle} \}. $$

(By the result quoted above, $U_2 = (G, 2)$, for instance.)

**Theorem 6.** There exist real numbers $\beta_n$ in $(1, 2)$ such that $U_n = (\beta_n, 2)$ for any $n \geq 2$. Furthermore, $\beta_n < \beta_m$ if and only if $n \prec m$ in the sense of the Sharkovskii ordering.

For a proof see [1]. Thus, once an $n$-cycle occurs at some $\beta$, it lives for any larger $\beta$. We have

$$ G = \beta_2 < \beta_4 < \beta_8 < \cdots < \beta' < \cdots < \beta_7 < \beta_5 < \beta_3. $$

There exists an explicit formula for the minimal polynomial for $\beta_n$ for any natural $n \geq 2$ (written as $n = 2^k(2\ell + 1)$) – see [1]. For the table of the first 8 values of $\beta_n$ see Table 2.1 below.

<table>
<thead>
<tr>
<th>$\beta_n$</th>
<th>period</th>
<th>minimal polynomial</th>
<th>numerical value</th>
<th>below $\beta'$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 2$</td>
<td>01</td>
<td>$x^2 - x - 1$</td>
<td>1.61803</td>
<td>yes</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>0110</td>
<td>$x^3 - 2x^2 + x - 1$</td>
<td>1.75488</td>
<td>yes</td>
</tr>
<tr>
<td>$n = 8$</td>
<td>0110 1001</td>
<td>$x^5 - 2x^4 + x^2 - 1$</td>
<td>1.78460</td>
<td>yes</td>
</tr>
<tr>
<td>$n = 6$</td>
<td>011010</td>
<td>$x^6 - x^5 - x^4 - x^2 - 1$</td>
<td>1.78854</td>
<td>no</td>
</tr>
<tr>
<td>$n = 7$</td>
<td>0110101</td>
<td>$x^7 - 2x^6 + x^4 - x^2 - 1$</td>
<td>1.80509</td>
<td>no</td>
</tr>
<tr>
<td>$n = 5$</td>
<td>01101</td>
<td>$x^7 - x^4 - x^3 - x - 1$</td>
<td>1.81240</td>
<td>no</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>011</td>
<td>$x^3 - x^2 - x - 1$</td>
<td>1.83929</td>
<td>no</td>
</tr>
</tbody>
</table>

**Table 2.1.** The table of $\beta_n$ for small values of $n$

Figure 7 indicates how this problem can be related to the classical one-dimensional setting.

More precisely, define the map $h : \{0, 1\}^N \to \{L, R\}^N$ as follows (* denotes an arbitrary – but fixed – tail):

- $h(0*) = Lh(*)$;
- $h(1^a0^b1*) = RL^{a-1}RL^{b-1}h(1*)$ for $a, b \geq 1$;
- $h(1^a0^\infty) = RL^{a-1}RL^\infty$;
- $h(1^\infty) = RL^\infty$.

Then $h$ is one-to-one and maps the orbits of the shift on the set of unique $\beta$-expansions into the orbits of $T_\beta$ which do not fall into $C$.

Let $\prec$ denote the standard *lexicographic order* on the sequences of 0s and 1s, namely, $\varepsilon \prec \varepsilon'$ if $\varepsilon_i \equiv \varepsilon'_i$, $1 \leq i \leq k$ and $\varepsilon_{k+1} < \varepsilon'_{k+1}$.
Let $\prec_u$ denote the unimodal order on the itineraries of $T_\beta$, i.e., $L \prec_u C \prec_u R$ and $\varepsilon \prec_u \varepsilon'$ if $\varepsilon_i \equiv \varepsilon'_i$, $1 \leq i \leq k$ and either $\varepsilon_{k+1} \prec_u \varepsilon'_{k+1}$ with $\#\{i \in [1, k] : \varepsilon_i = R\}$ even or $\varepsilon_{k+1} \succ_u \varepsilon'_{k+1}$ with $\#\{i \in [1, k] : \varepsilon_i = R\}$ odd.

We have for $\varepsilon, \varepsilon' \in \Sigma$,

$$\varepsilon \prec \varepsilon' \iff h(\varepsilon) \prec_u h(\varepsilon').$$

The map $h$ helps to prove our version of the Sharkovskii theorem via the classical one.

2.1. Finite number of beta-expansions. Put

$$B_m = \{ \beta \in (G, 2) : \exists x \in [0, 1/(\beta - 1)] \text{which has exactly } m \text{ expansions in base } \beta \}.$$ 

Lemma 7. We have $B_m \subset B_2$ for $m \geq 3$ and $m \in \mathbb{N}$.

Hence if $\beta \notin B_2$, then we have the following dichotomy: either a number $x \in J_\beta$ has a unique $\beta$-expansion or infinitely many of them.
Theorem 8 (N. Sidorov, 2009). The smallest element of $B_2$ is $\tilde{\beta}_2$, the appropriate root of $x^4 = 2x^2 + x + 1$, with the numerical value $\tilde{\beta}_2 \approx 1.71064$. Furthermore, $B_2 \cap (\tilde{\beta}_2, \beta_4) = \emptyset$.

Here, as above, $\beta_4 \approx 1.75488$ is the appropriate root of $x^3 = 2x^2 - x + 1$.

**Theorem 9.** For $\beta \in (G, \beta')$ the strong dichotomy holds provided $\beta$ is transcendental.

(Strong dichotomy means that any $x$ has either a unique $\beta$-expansion or a continuum of them.)

So, we know that $B_2 \cap (G, \beta')$ is countable (lower order).

**Theorem 10** (middle order). The set $B_2 \cap (\beta', \beta' + \delta)$ has the cardinality of the continuum for any $\delta > 0$.

**Theorem 11** (top order). Let, as above, $\beta_3$ denote the root of $x^3 = x^2 + x + 1$, $T \approx 1.83929$. Then $[\beta_3, 2) \subset B_2$, i.e., there always exists $x$ which has exactly two $\beta$-expansions provided $\beta \geq \beta_3$.

A similar result holds for $B_m$ for any $m \geq 3$.

3. **Topology of sums in nonnegative powers of $\beta > 1$**

Let $1 < \beta < 2$ be our parameter. Put

$$\Lambda_n(\beta) = \left\{ \sum_{k=0}^{n} a_k \beta^k \mid a_k \in \{-1, 0, 1\} \right\}$$

and

$$\Lambda(\beta) = \bigcup_{n \geq 1} \Lambda_n(\beta).$$

Trivial properties of $\Lambda(\beta)$:
- countable;
- unbounded;
- symmetric about 0;

**Question:** what is the topology of $\Lambda(\beta)$? Is it dense? discrete? neither?

**Theorem 12** (Garsia, 1962 [13]). Let $\beta$ be a Pisot number, i.e., an algebraic integer whose other conjugates are less than 1 in modulus. Then $\Lambda(\beta)$ is uniformly discrete.

**Proof.** Without loss of generality we may assume $x, y \in \Lambda_n(\beta)$ and $x \neq y$. Then $x - y = \sum_{k=0}^{n} \varepsilon_k \beta^k$ with $\varepsilon_k \in \{-2, -1, 0, 1, 2\}$. Put

$$P(t) = \sum_{k=0}^{n} \varepsilon_k t^k.$$ 

Let $\beta_1 = \beta, \beta_2, \ldots, \beta_d$ be the conjugates of $\beta$. Since $P(\beta) \neq 0$, we have $P(\beta_j) \neq 0$ for all $j$. Hence $\prod_{j} P(\beta_j) \neq 0$. As this product is an integer (exercise!), we have

$$\left| \prod_{j} P(\beta_j) \right| \geq 1.$$
Consequently,
\[ |P(\beta)| \geq \frac{1}{\prod_{j \geq 2} P(\beta_j)}. \]

Since \(|\beta_j| < 1\) for all \(j \geq 2\) (Pisot!), we have
\[ \sum_{i=0}^{n} \varepsilon_i \beta_i^j = O(1), \]
whence \(|P(\beta)| \geq \text{const.} \)  \(\Box\)

**Theorem 13** (folklore). If \(\beta\) is transcendental, then 0 is a limit point of \(\Lambda(\beta)\).

**Proof.** Put
\[ D_n(\beta) = \left\{ \sum_{k=0}^{n} a_k \beta^k \mid a_k \in \{0, 1\} \right\}. \]

Since \(\beta\) is transcendental, \(z_n(\beta) := \#D_n(\beta) = 2^{n+1}\). On the other hand, \(\max D_n(\beta) = O(\beta^n) \ll 2^n\).

By the pigeonhole principle, there exist \(x, y \in D_n(\beta)\) such that
\[ |x - y| \leq \text{const} \cdot \left( \frac{\beta}{2} \right)^n = o(1). \]

Since \(x - y \in \Lambda_n(\beta)\), we are done. \(\Box\)

**Theorem 14** (Drobot, 1973 [6]). If 0 is a limit point of \(\Lambda(\beta)\), then \(\Lambda(\beta)\) is dense in \(\mathbb{R}\).

Thus, if \(\beta\) is not of height 1 (i.e., is not a root of \(-1, 0, 1\) polynomial), then \(\Lambda(\beta)\) is dense. (For example, \(\beta = \sqrt{2}\).)

**Conjecture.** If \(\beta\) is not Pisot, then \(z_n(\beta) \gg \beta^n\) and consequently, \(\Lambda(\beta)\) is dense.

**Definition 15.** We say that an algebraic \(\beta > 1\) is a Perron number if \(|\alpha| < \beta\) for any conjugate \(\alpha\) of \(\beta\).

**Theorem 16** (Sidorov and Solomyak, 2009 [21]). If \(\beta\) is not Perron, then \(\Lambda(\beta)\) is dense in \(\mathbb{R}\).

**Proof.** Here is a crude idea of our proof: assume there exists \(\alpha\) which is a conjugate of \(\beta\) such that \(\beta < |\alpha|\). It is easy to see that \(z_n(\beta) = z_n(\alpha)\) (since there is a natural bijection between the sets \(D_n(\beta)\) and \(D_n(\alpha)\)). Then we show that \(z_n(\alpha) \geq \text{const} \cdot |\alpha|^n\) (this is the key point of our proof), whence \(z_n(\beta) \gg \beta^n\), and we apply the pigeonhole principle. \(\Box\)

Let \(D(\beta)\) denote the set of all finite 0-1 sums in nonnegative powers of \(\beta\), i.e., \(D(\beta) = \bigcup_{n \geq 1} D_n(\beta)\). Since for any \(E > 0\) we have that \([0, E] \cap D(\beta)\) is finite, \(D(\beta)\) is discrete.

Write
\[ D(\beta) = \{ y_0(\beta) < y_1(\beta) < \ldots \}. \]
Put
(3.1) \( \ell(\beta) = \lim \inf_n (y_{n+1} - y_n) \)
and
\[ L(\beta) = \lim \sup_n (y_{n+1} - y_n). \]

It is obvious that \( \ell(\beta) = 0 \) if and only if 0 is a limit point of \( \Lambda(\beta) \). Hence \( \ell(\beta) = 0 \iff \Lambda(\beta) \) is dense in \( \mathbb{R} \).

**Theorem 17** (Erdős and Komornik, 1998 [10]). For any \( \beta < 2^{1/4} \) we have \( L(\beta) = 0 \).

It is also known that \( L(\sqrt{2}) = 0 \) and \( L(\beta) = \beta \) for any \( \beta \geq \frac{1+\sqrt{5}}{2} \) (see Problem Sheet 2).

No \( \beta \in \left( \sqrt{2}, \frac{1+\sqrt{5}}{2} \right) \) with \( L(\beta) = 0 \) is known.

4. **Bernoulli convolutions**

Let \( \beta > 1 \) and define the Bernoulli convolution \( \xi_\beta \) as follows. Let \( b_n(\beta) \) be the two-point distribution such that \( b_n(-\beta^{-n}) = b_n(\beta^{-n}) = 1/2 \). Now
\[ \xi_\beta = b_1(\beta) * b_2(\beta) * \ldots, \]
an infinite convolution. Note that \( b_1(\beta) * b_2(\beta) * \ldots * b_n(\beta) \) is supported by the finite set \( \{ \sum_{k=1}^n \epsilon_k \beta^{-k} : \epsilon_k \in \{ -1, 1 \} \} \) and each point has the measure \( 2^{-n} \). (Some of them may coincide if \( \beta \) is algebraic.) Hence for any Borel set \( E \subset \mathbb{R} \),
\[ \xi_\beta (E) = \mathbb{P} \left\{ (a_1, a_2, \ldots) \in \{ -1, 1 \}^\mathbb{N} : \sum_{k=1}^\infty a_k \beta^{-k} \in E \right\}, \]
where \( \mathbb{P} \) is the product measure on \( \{ -1, 1 \}^\mathbb{N} \) with \( \mathbb{P}(a_1 = -1) = \mathbb{P}(a_1 = 1) = 1/2 \).

The reason people have got interested in Bernoulli convolutions in the 1930s (see [23] for a comprehensive survey) is their especially nice Fourier transform:
\[ \hat{\xi}_\beta (x) = \prod_{n=1}^\infty \frac{1}{2} \left( e^{-i\beta^{-n} x} + e^{i\beta^{-n} x} \right) = \prod_{n=1}^\infty \cos(\beta^{-n} x). \]

We also define the measure \( \nu_\beta \) in a similar way (replacing \(-1\) with \(0\)):
\[ \nu_\beta (E) = \mathbb{P} \left\{ (a_1, a_2, \ldots) \in \{ 0, 1 \}^\mathbb{N} : \sum_{k=1}^\infty a_k \beta^{-k} \in E \right\}. \]

In other words, \( \nu_\beta \) “measures” how many \( \beta \)-expansions fall into a given set. It is easy to see that \( \nu_\beta \) is a scaled copy of \( \xi_\beta \) (exercise!), so their important properties should be the same.
Recall that a measure $\nu$ is called \textit{absolutely continuous} (with respect to the Lebesgue measure $\mathcal{L}$) if $\mathcal{L}(E) = 0$ implies $\nu(E) = 0$. In this case there exists an integrable function $h$ (the Radon-Nikodym density) such that $\nu(E) = \int_E h(x) \, dx$.

A measure $\nu$ is called \textit{singular} if there exists a Borel set $F$ such that $\nu(F) = 0$ and $\mathcal{L}(F) = 1$. (Here $\mathcal{L}$ is a probability measure.)

\textbf{Theorem 18} (Jessen-Wintner, 1935). \textit{For any $\beta > 1$ the measure $\nu_\beta$ is either absolutely continuous or singular.}

This result is often referred to as the \textit{Law of Pure Types}.

Note that if $\beta = 2$, then $\nu_\beta$ is none other than the Lebesgue measure. If $\beta > 2$, then $\nu_\beta$ “sits” on a Cantor set of zero Lebesgue measure (exercise!) and hence is singular. But what happens if $\beta \in (1,2)$?

\textbf{Definition 19}. An algebraic integer $\beta > 1$ is called a \textit{Pisot number} (or a Pisot-Vijayaraghavan (PV) number) if all its other Galois conjugates are less than 1 in modulus.

The set of Pisot numbers is known to be closed (sic!). The smallest Pisot number is the real root of $x^3 - x - 1$. The smallest limit point of the set of Pisot numbers is the golden ratio. The main property of a Pisot number $\beta$ is that there exists a sequence of positive integers $z_N$ such that

\begin{equation}
\beta^N = z_N + O(\gamma^N), \quad N \to +\infty
\end{equation}

for some $\gamma \in (0, 1)$.

Recall the Riemann-Lebesgue Lemma (or Theorem in some textbooks): for any $f$ in $L^1(\mathbb{R})$ we have $\hat{f}(x) \to 0$ as $x \to \pm \infty$. Consequently, for any absolutely continuous measure $\nu$ we have $\hat{\nu}(x) \to 0$ as $x \to \pm \infty$.

\textbf{Theorem 20} (Erdős, 1939 [7]). \textit{For any Pisot $\beta \in (1,2)$ the Bernoulli convolution $\xi_\beta$ is singular.}

\textbf{Proof}. We will show that $\hat{\xi}_\beta(x) \neq 0$ as $x \to +\infty$, which will imply that $\xi_\beta$ cannot be absolutely continuous. Therefore, by the Law of Pure Types, it must be singular.

Put $x_N = 2\pi\beta^N$. We have

$$
\hat{\xi}_\beta(x_N) = \prod_{n=1}^{\infty} \cos(2\pi\beta^{-n} x) \\
= \cos(2\pi\beta^N) \cdot \cos(2\pi\beta^{N-1}) \cdots \cos(2\pi\beta) \cdot \hat{\xi}_\beta(2\pi).
$$

Since $\beta$ is irrational, $\hat{\xi}_\beta(2\pi) \neq 0$ (check it!). In view of (4.1), $\cos(2\pi\beta^k) = \cos(2\pi\beta^k - 2\pi z_k) = 1 - O(\gamma^k)$. Hence

$$
|\cos(2\pi\beta^N) \cdot \cos(2\pi\beta^{N-1}) \cdots \cos(2\pi\beta)| \geq \text{const},
$$

whence

$$
|\hat{\nu}_\beta(x_N)| \geq \text{const}'.
$$

\boxend
There exists an alternative proof [19] in which we construct a measure \( \tilde{\nu}_{\beta} \) which is equivalent to \( \nu_{\beta} \) such that the greedy \( \beta \)-transformation preserves it, and it is ergodic.

**Theorem 21** (B. Solomyak, 1995 [22]). For Lebesgue-a.e. \( \beta \in (1, 2) \) the Bernoulli convolution \( \xi_{\beta} \) is absolutely continuous.

There is only one explicit family of \( \beta \) for which it is known that \( \xi_{\beta} \) is absolutely continuous.

**Definition 22.** An algebraic integer \( \beta > 1 \) is called a Garsia number if all its Galois conjugates are greater than 1 in modulus, and the constant term of its minimal polynomial is \( \pm 2 \).

Such is \( \sqrt{2} \) or the appropriate root of \( x^4 - x - 2 \), say.

**Theorem 23** (Garsia, 1962 [13]). For any Garsia \( \beta \) the Bernoulli convolution \( \xi_{\beta} \) is absolutely continuous with a bounded density.

5. Multidimensional \( \beta \)-expansions

Let, as above, \( \beta > 1 \) be our parameter. Consider a pair of maps (similitudes) in the real line:

\[
\begin{align*}
  f_0(x) &= \frac{x}{\beta}, \\
  f_1(x) &= \frac{x}{\beta} + 1.
\end{align*}
\]

They constitute an iterated function system (IFS). That is, choose 0 as a starting point, and for any sequence \( (\varepsilon_1, \varepsilon_2, \ldots) \) of 0s and 1s:

\[
x = \lim_{N \to +\infty} f_{\varepsilon_1} \cdots f_{\varepsilon_N}(0).
\]

The set of all \( x \)'s that are representable in such a form, is called the invariant set \( I_{\beta} \) of the IFS.

Unlike a general IFS (see, e.g., [11]), in our model this expression can be given in a very simple form:

\[
f_{\varepsilon_1} \cdots f_{\varepsilon_N}(0) = \beta^{-1} \varepsilon_1 + \beta^{-1}(\varepsilon_2 + \beta^{-1}(\varepsilon_3 + \cdots + \beta^{-1} \varepsilon_N) \ldots))
\]

\[
= \sum_{n=1}^{N} \varepsilon_n \beta^{-n},
\]

whence

\[
x = \lim_{N} \sum_{n=1}^{N} \varepsilon_n \beta^{-n} = \sum_{n=1}^{\infty} \varepsilon_n \beta^{-n}.
\]

We see that the invariant set is none other than the set of \( \beta \)-expansions.

Let \( \mathbf{p}_0, \ldots, \mathbf{p}_k \) now be points in \( \mathbb{R}^d \). Consider the IFS – a general collection of similitudes:

\[
(5.1) \quad f_i(x) = \beta^{-1} x + (1 - \beta^{-1}) \mathbf{p}_i.
\]
Then any point $x$ in the invariant set has a representation in the form

$$x = (\beta - 1) \sum_{n=1}^{\infty} \beta^{-n} a_n,$$

where $a_n$ is one of the vertices $p_i$.

Unlike the one-dimensional case, the invariant set $J_\beta$ (which lies in the convex hull of the set $\{p_0, \ldots, p_k\}$) may have a complicated structure.

Let $p_0, p_1, p_2$ be the vertices of a triangle $\Delta$ in $\mathbb{R}^2$ (equilateral, say—this does not matter!). Note first that if $\beta \leq 3/2$, then $J_\beta = \Delta$. If $\beta \in (3/2, 2)$, then we have both holes and overlaps.

The most famous case is $\beta = 2$ – see Figure 8. Its Hausdorff dimension is known to be equal to $\log 3 / \log 2$.

Assume now $\beta \in (3/2, 2)$. Let first $\beta = \frac{1+\sqrt{5}}{2}$. We get the following nice fractal – see Figure 9.

**Theorem 24** (D. Broomhead, J. Montaldi and N. Sidorov, 2003 [3]). The invariant set $J_\beta$ is **totally self-similar**, i.e.,

$$f_{\varepsilon_0} \cdots f_{\varepsilon_{n-1}}(J_\beta) = f_{\varepsilon_0} \cdots f_{\varepsilon_{n-1}}(\Delta) \cap J_\beta$$

for any $\varepsilon_0, \ldots, \varepsilon_{n-1}$. 

**Figure 8.** The Sierpiński Gasket
Theorem 25 (D. Broomhead, J. Montaldi and N. Sidorov, 2003 [3]).

$$\dim_H(J_\beta) = -\frac{\log \tau}{\log \beta} = 1.93063 \ldots,$$

where \( \tau \approx 0.39493 \) is a root of the polynomial \( 3z^3 - 3z + 1 \), namely,

$$\tau = \frac{2}{\sqrt{3}} \cos(\pi/18).$$

Theorem 26 (D. Broomhead, J. Montaldi and N. Sidorov, 2003 [3]). If the invariant set \( J_\beta \) is totally self-similar for some \( \beta \in (3/2, 2) \), then \( \beta \) satisfies

$$\beta^m = \beta^{m-1} + \beta^{m-2} + \cdots + \beta + 1$$

for some \( m \geq 2 \) (multinacci numbers).

Here is a sketch of the proof of the key Theorem 24 (for an arbitrary multinacci \( \beta \)). Let \( x, y, z \) be the distances to the sides of \( \Delta \) so that \( x + y + z = 1 \). These are called barycentric coordinates.

Then the \( f_i \) are linear maps in barycentric coordinates, and one can easily check that
\begin{align*}
 f_0 &= \begin{pmatrix} 1 & 1 - \lambda & 1 - \lambda \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \\
 f_1 &= \begin{pmatrix} \lambda & 0 & 0 \\ 1 - \lambda & 1 & 1 - \lambda \\ 0 & 0 & \lambda \end{pmatrix}, \\
 f_2 &= \begin{pmatrix} \lambda & 0 & 0 \\ 1 - \lambda & 1 - \lambda & 1 \end{pmatrix},
\end{align*}

where \( \lambda = \beta^{-1} \). Moreover,

\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lim_{N \to +\infty} f_{\varepsilon_0} \cdots f_{\varepsilon_N}(0)
\]

\[
= \begin{pmatrix} (\beta - 1) \sum_{k=1}^{\infty} a_k \beta^{-k} \\ (\beta - 1) \sum_{k=1}^{\infty} b_k \beta^{-k} \\ (\beta - 1) \sum_{k=1}^{\infty} c_k \beta^{-k} \end{pmatrix},
\]

where \( a_k, b_k, c_k \in \{0, 1\} \) and \( a_k + b_k + c_k = 1 \). (In fact, \( a_k = \chi_{\{\varepsilon_k = 0\}}, b_k = \chi_{\{\varepsilon_k = 1\}}, c_k = \chi_{\{\varepsilon_k = 2\}} \).) Let \( \Delta_0 = \Delta \), and

\[
\Delta_n = \bigcup_{i=0}^{2} f_i(\Delta_{n-1}), \quad n \geq 1.
\]

The central hole \( H_0 := \Delta \setminus \Delta_1 \). Then each hole is a subset of an image of \( H_0 \).

The key to the proof is the fact that for the multinacci \( \beta \) any image of the central hole is a hole. This is easily equivalent to the total self-similarity of \( J_\beta \).

It suffices to show that \( H_n := f_{\varepsilon_0} \cdots f_{\varepsilon_{n-1}}(H_0) \) has an empty intersection with \( \Delta_{n+1} \).

This is equivalent to the fact that the system

\[
\beta^{-n-1} + \sum_{k=1}^{n-1} a_k \beta^{-k} > \sum_{k=1}^{n} a_k' \beta^{-k},
\]

\[
\beta^{-n-1} + \sum_{k=1}^{n-1} b_k \beta^{-k} > \sum_{k=1}^{n} b_k' \beta^{-k},
\]

\[
\beta^{-n-1} + \sum_{k=1}^{n-1} c_k \beta^{-k} > \sum_{k=1}^{n} c_k' \beta^{-k}
\]

does not have a solution. This in turn follows from

**Theorem 27** (P. Erdős, I. Joó, M. Joó, 1992 [8]). Let \( \ell(\beta) \) be given by (3.1). Then \( \ell(\beta) = \beta^{-1} \) if \( \beta \) is a multinacci number.
In other words, $\beta^{-1}$ is the exact separation constant in the Garsia separation lemma (Theorem 12) if $\beta$ is multinacci. See Figure 10 for the set of uniqueness for the golden gasket.

The main problem remaining is to determine for which $\beta$ the attractor $J_\beta$ has positive two-dimensional Lebesgue measure and for which zero Lebesgue measure.

**Theorem 28.** [3] For $\beta$ sufficiently close to $3/2$ the measure is positive and, moreover, the interior of $J_\beta$ is nonempty. For $\beta > \sqrt{3}$ the measure of $J_\beta$ is zero.

The numerics suggests the following

**Conjecture.** (1) For each $\beta \in (\frac{3}{2}, \frac{1+\sqrt{5}}{2})$ the attractor $J_\beta$ has a nonempty interior – see Figure 11.
(2) For each $\beta \in (\frac{1+\sqrt{5}}{2}, \sqrt{3})$ it has an empty interior – see Figure 12.

Return to the general setting (5.1). There exists an analogue of Theorem 1:

**Theorem 29** (Sidorov, 2007 [20]). For each $p_0, \ldots, p_{m-1}$ there exists $\beta_0 > 1$ such that for any $\beta > \beta_0$,

1. There are no holes in $J_\beta$.
2. Each point $x$ in the convex hull of $\{p_0, \ldots, p_{m-1}\}$ except when $x$ is $p_i$, has $2^{\aleph_0}$ distinct addresses.

Thus, $\beta_0$ in this theorem is a direct analogue of the golden ratio in the one-dimensional setting. To determine the sharp value of $\beta_0$ for a given collection $\{p_0, \ldots, p_{m-1}\}$ is an interesting problem.
Figure 11. The invariant set $J_\beta$ for $\beta = 1.54$

Figure 12. The invariant set $J_\beta$ for $\beta = 1.69$

There also exists a multidimensional generalization of Theorem 3:

**Theorem 30.** [20] Assume that the attractor $J_\beta$ has no holes plus some technical condition. Then Lebesgue-a.e. $x$ in the convex hull of the $p_i$ has a continuum of distinct $\beta$-expansions, and the exceptional set has Hausdorff dimension strictly less than $d$, the dimension of the convex hull of the $p_i$. 
References

[22] B. Solomyak, On the random series $\sum \pm \lambda^i$ (an Erdos problem), Annals of Math. 142 (1995), 611-625.