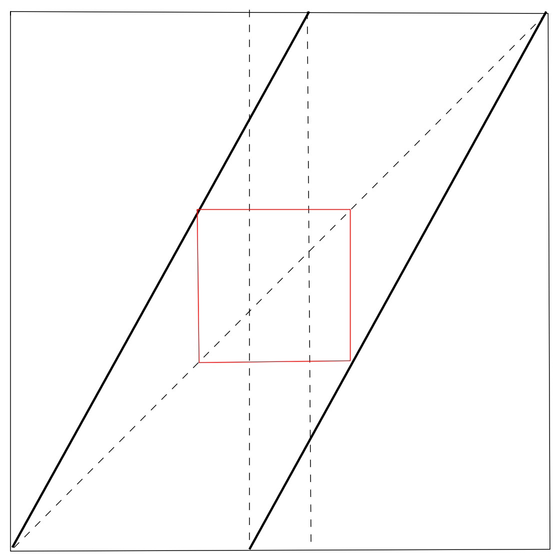

Figure 1: The 2-cycle in Problem 3

Nikita Sidorov (Manchester)

Mini-course: 'Expansions in non-integer bases'

Problem Sheet 1

1. Let $\beta = \frac{1+\sqrt{5}}{2}$ and $x = 1$. Find all β-expansions of x. (*Hint: use the fact that* $\beta^{-1} + \beta^{-2} = 1$.)

2. Assume $1 < \beta < \frac{1+\sqrt{5}}{2}$ and $0 < x < \frac{1}{\beta-1}$. Prove accurately that x has a continuum of distinct β-expansions.

3. Let $\frac{1+\sqrt{5}}{2} < \beta < 2$ and Find $x < \frac{1}{\beta}$ such that $T_\beta(x) > \frac{1}{\beta(\beta-1)}$ and $T_\beta^2(x) = x$. (Hence x has a unique β-expansion.)

4. Show that if some x has exactly m β-expansions ($m \geq 3$), then there exists x' which has exactly two β-expansions.

5. Let $\tau_\beta : [0,1) \to [0,1)$ be defined as $\tau_\beta(x) = \beta x \bmod 1$. (The β-transformation.) Suppose $\beta = \frac{1+\sqrt{5}}{2}$; prove that the absolutely continuous measure with the density

$$h(x) = \begin{cases} \frac{5+3\sqrt{5}}{2}, & 0 \leq x < \frac{\sqrt{5}-1}{2} \\ \frac{5+\sqrt{5}}{2}, & \frac{\sqrt{5}-1}{2} < x < 1 \end{cases}$$

is τ_β-invariant. (Rényi, 1957)