The Mean-Median Map

Jonathan Hoseana and Franco Vivaldi
Take any finite sequence of real numbers:

\[(11, 18, 23)\].

Adjoin to it a new number in such a way that the mean of the resulting sequence is equal to the median of the original sequence:

\[
11 + 18 + 23 + x = 18 \Rightarrow x = 20.
\]

So we now have \[(11, 18, 23, 20)\].

Do this recursively:

\[(11, 18, 23, 20, 23, 25, 26, 27, \ldots)\]

The mean-median sequence of \((11, 18, 23)\) stabilises!

Strong terminating conjecture [Schultz & Shiflett, 2005]
The mean-median sequence of any initial real sequence is eventually constant.
Take any finite sequence of real numbers:

(11, 18, 23).

Do this recursively:

(11, 18, 23, 20, 23, 25, 23, 23, 23, ...)

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Take any finite sequence of real numbers:

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Adjoin to it a new number in such a way that the mean of the resulting sequence is equal to the median of the original sequence:

$11 + 18 + 23 + x / 4 = 18$  \[\Rightarrow x = 20\]

So we now have

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$(11, 18, 23, 20, 23, 25, \ldots)$

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Adjoin to it a new number in such a way that the mean of the resulting sequence is equal to the median of the original sequence:

\[
\frac{11 + 18 + 23 + x}{4} = 18
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So we now have \((11, 18, 23, 20)\).

Do this recursively:

\((11, 18, 23, 20, 23, 25, \ldots, 61, 67, \ldots, 23, 23, 23, \ldots)\)

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Several conjectures have been formulated on mean-median sequences: they are all open.
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The map:

\[ x_{n+1} = (n + 1)M_n - S_n \]

n-th median \( n \)-th sum
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▶ A recursion whose order grows with the iteration.

1. Each new term is the difference of two diverging quantities.
2. The median is a non-smooth function of the data.
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Simplest non-trivial initial sequence

\[(a, b, c) \quad a \leq b \leq c\]
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\[(a, b, c)\] is affine-equivalent to \((0, x, 1)\).

\[O\] \(\longleftrightarrow\) \(O\)

\[a \quad b \quad c\]

\[0 \quad x \quad 1\]
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It suffices to study \((0, x, 1), \quad 0 \leq x \leq 1\).

[Chamberland & Martelli, 2007]: \(\frac{1}{2} \leq x \leq \frac{2}{3}\) suffices.
A typical orbit \[(0, x, 1)\] \[\frac{1}{2} \leq x \leq \frac{2}{3}\]
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\[x = \frac{71}{128}\]
A typical orbit \((0, x, 1)\) \(\frac{1}{2} \leq x \leq \frac{2}{3}\)

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The sequence \(M_n\) of medians is locally monotonic (non-decreasing for \(1/2 \leq x \leq 2/3\)).
A typical orbit $(0, x, 1)$ with $\frac{1}{2} \leq x \leq \frac{2}{3}$

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If $M_{n+1} = M_n$, then the sequence stabilises.
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Convergence without stabilisation has never been observed.
Invariant modules

Let $A$ be the $\mathbb{Z}[\frac{1}{2}]$-module generated by the initial sequence $(x_1, \ldots, x_n)$.

$A$ is invariant under the mean-median map.

For a rational initial sequence, we have $A = \mathbb{1}_d \mathbb{Z}[\frac{1}{2}]$, where $d$ is the largest odd divisor of the lcm of the denominators of the $x_i$'s.

The module $A$ is a layered space $\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots k/d \cdots k/2d \cdots k/4d \cdots k/5d \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$.

How deep do orbits sink?
Invariant modules

- $\mathbb{Z}\left[\frac{1}{2}\right]$: the ring of rationals whose denominator is a power of 2.

Let $A$ be the $\mathbb{Z}\left[\frac{1}{2}\right]$-module generated by the initial sequence $(x_1,...,x_n)$.

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For a rational initial sequence, we have $A = \frac{1}{d}\mathbb{Z}\left[\frac{1}{2}\right]$, where $d$ is the largest odd divisor of the lcm of the denominators of the $x_i$s.

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$$
\begin{align*}
&\cdots \\
&\frac{k}{d} \\
&\frac{k}{2d} \\
&\frac{3k}{2d} \\
&\frac{5k}{2d} \\
&\vdots \\
&\frac{d^r k}{2^d r} \\
\end{align*}
$$

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\[
\begin{align*}
\cdots & \quad \cdots k/d \\
\cdots & \quad \cdots k/2d \\
\cdots & \quad \cdots k/2^2d \\
\cdots & \quad \cdots k/2^3d \\
\cdots & \quad \cdots k/2^4d \\
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\end{align*}
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How deep do orbits sink?
The $n$-th effective exponent $\kappa(n)$: the largest exponent of 2 in the denominators of $x_1, \ldots, x_n$. 

Substantial cancellations slow down the growth of $\kappa(n)$. 

The trivial estimate $\kappa(n) \leq \lfloor n/2 \rfloor$ is unhelpful.
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![Graph showing the growth of $\kappa(n)$ over $n$]
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![Graph showing the growth of $\kappa(n)$ compared to $\frac{n}{2}$ with stabilization occuring over $\mathbb{Q}$.]
From numbers to piecewise-affine functions

\[ Y_1(x) = 0 \]
\[ Y_2(x) = x \]
\[ Y_3(x) = 1 \]

We view the initial sequence \((0, x, 1)\) as a sequence of affine functions.
From numbers to piecewise-affine functions

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\[ Y_h(x) \]

\[ M_3 \]
From numbers to piecewise-affine functions

\[ Y_h(x) \]

\[ \mathcal{M}_4 \]
From numbers to piecewise-affine functions
From numbers to piecewise-affine functions

$M_6$
From numbers to piecewise-affine functions

\[ Y_n(x) \]

\[ M_7 \]
From numbers to piecewise-affine functions

$Y_n(x)$

$\mathcal{M}_8$
From numbers to piecewise-affine functions

\[ Y_n(x) \]

How ubiquitous are the singularities?
From numbers to piecewise-affine functions

How ubiquitous are the singularities?
The limit function $m(x) = \lim_{n \to \infty} M_n(x)$
The limit function \( m(x) = \lim_{n \to \infty} \mathcal{M}_n(x) \)
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Conjecture [Chamberland & Martelli, 2007]

The limit function of \((0, x, 1)\) is continuous.
The limit function: what do we know?
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Near 1/2 [Schultz & Shiflett, 2005]
The limit function: what do we know?

- Near 1/2 [Schultz & Shiflett, 2005]
- Near all fractions with denominator at most 18 (computer-assisted proof) [Cellarosi & Munday, 2016]
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Limit function vs. Takagi function
Limit function vs. Takagi function

\[ m(x) = M_3 + \sum_{n=4}^{\infty} \Delta M_n \]

\[ \Delta M_n = M_n - M_{n-1} \]

\[ T(x) = \sum_{n=0}^{\infty} \frac{[2^n x]}{2^n} \]

\[ [x] = \min \{|x - n| : n \in \mathbb{Z}\} \]
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Continuous, unbounded variation everywhere, 1-D [Takagi, 1903]
Limit function vs. Takagi function

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Continuous, unbounded variation everywhere, 1-D [Takagi, 1903]
The limit function near an X-point
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an X-point
The limit function near an X-point

auxiliary function

an X-point of rank 1
The limit function near an X-point

auxiliary function of multiplicity 2

an X-point of rank 2
The limit function near an X-point

\[ M_{\tau(p)-1} \]
The limit function near an X-point

\[ M_{\tau(p) - 1} \]
The limit function near an X-point

\[
\mathcal{M}_{\tau(p)}
\]
The limit function near an X-point

\[ M_{\tau(p)+1} \]
The limit function near an X-point

\[ M_{\tau(p)+2} \]
The limit function near an X-point
The limit function near an X-point

\[ M_{\tau(p)+4} \]
The limit function near an X-point

\[ M_{\tau(p)+5} \]
The limit function near an X-point

\[ \mathcal{M}_{\tau(p)+6} \]
The limit function near an X-point

The local shape of the limit function depends on whether the interval of regularity of the median remains finite or shrinks to $p$. 

$p$
Theorem. The limit function near an X-point of rank at least 2:

\[ \Rightarrow \text{either} \quad \text{or} \]

In our computations, we never encountered the second scenario.

Theorem. The limit function near an X-point of rank 1:

Under suitable conditions which hold in most cases.
Theorem. *The limit function near an X-point of rank at least 2:*

\[ \Rightarrow \begin{cases} \text{either} \\ \text{or} \end{cases} \]

In our computations, we never encountered the second scenario.
Theorem. *The limit function near an X-point of rank at least 2:*

\[
\begin{align*}
\Rightarrow \text{ either } & \quad \text{ or } \\
\end{align*}
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In our computations, we never encountered the second scenario.

Theorem. *The limit function near an X-point of rank 1:*

\[
\begin{align*}
\Rightarrow \text{ under suitable conditions which hold in most cases } \\
\end{align*}
\]
Symmetry near an X-point
Symmetry near an X-point

Recall:

\[ \text{Diagram of symmetry near an X-point} \]
Symmetry near an X-point

Recall:

\[ O \]

Aim: Establish this near an X-point.
Symmetry near an X-point

Recall:

Aim: Establish this near an X-point.
Symmetry near an X-point

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Symmetry near an X-point

Recall:

Aim: Establish this near an X-point.

Q: Does the limit function also obey this symmetry?
Symmetry near an X-point

Recall:

Aim: Establish this near an X-point.

Q: Does the limit function also obey this symmetry?

Theorem

If $Y_{\tau(p)}$ is an affine combination of $Y$, $\min\{Y_i, Y_j\}$, and $\max\{Y_i, Y_j\}$, then all functions $Y_n$, $n \geq \tau(p)$, and hence the limit, obey the symmetry.
Normal form near an X-point
Normal form near an X-point

\[ M_{t}^{-1} \]
Normal form near an X-point

\[ Y_j \]

\[ M_{t-1} \]

\[ Y_i \]
Normal form near an X-point
Normal form near an X-point
Normal form near an X-point
Normal form near an X-point

Sufficiently close to a stabilised X-point, the dynamics is largely independent from pre-stabilisation data.
Normal form near an X-point

- A one-parameter family of dynamical systems involving only the functions passing through the X-point.

\[ Y_{t+1}, Y_t, Y_i, Y_j \]

\[ p \]

\[ z_i := \frac{Y_i'}{\rightarrow 0} \quad \text{and} \quad z_i := \frac{Y_j'}{\rightarrow 1}. \]
Normal form near an X-point

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- If this sequence stabilises, then the mean-median map stabilises in a small neighbourhood of all X-points with the given stabilisation time.
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\[ \begin{array}{c}
\text{z}_n \\
\uparrow \\
t \quad \text{regular phase} \quad N_t \quad \text{irregular phase} \quad \tau_t \\
\end{array} \]

- Regular phase: \((t \leq n \leq N_t \sim t\sqrt{5})\)

\[
z_{t+4\ell+k} = \begin{cases} 
\frac{\ell+1}{2} t + \ell^2 + \ell & \text{if } k = 0 \\
\frac{\ell+1}{2} t + \ell^2 + \ell + 1 & \text{if } k = 1 \\
\frac{5}{4} t^2 + \frac{5}{2} \ell t + 5\ell^2 - \ell & \text{if } k = 2 \\
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Slow growth of denominators: \(\{z_t, \ldots, z_{N_t}\} \subseteq \frac{1}{2^2} \mathbb{Z}\).
Stabilisation over $\mathbb{Q}$

Conjecture

The stabilisation time of the orbit of $(0, x, 1)$ is unbounded for $x \in \mathbb{Q}$.

What could go wrong?

Fact (a regular system)

The functional orbit of $(0, x, 1, 1)$ consists of 63 distinct functions. In particular, the stabilisation time is bounded over $\mathbb{R}$.

Theorem

There is a family $\xi_i$ of rational initial sequences of increasing length, whose orbits have stabilisation time $\sim |\xi_i|^{2/3}$. 

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Using the normal form, we establish the strong terminating conjecture in specified neighbourhoods of $2^{791}$ fractions, improving [Cellarosi & Munday, 2016] by two orders of magnitude.

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![Diagram of a regular domain](image-url)
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\[
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\]

Conjecture
The Hausdorff dimension of the graph of the limit function is greater than 1.
A hierarchy of rationals

\[ m(x) \]
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Thank you for your attention
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▶ J. Hoseana, The mean-median map, MSc thesis, Queen Mary University of London (2015).