MTH 6107 Chaos and Fractals: Coursework 6

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This coursework will not be assessed

CONTENT: Conjugacy. Fractals.

Problem 1. What is chaos? Write a 100-word essay, using no mathematical symbols.

Problem 2. Write $f \sim g$ if f is topologically conjugate to g. Prove that $\sim$ is an equivalence relation, i.e., $\sim$ is reflexive, symmetrical and transitive.

Problem 3. Let f and g be conjugate via a diffeomorphism h. Let f have a fixed point at x^*.

(a) Prove that the multiplier of g at $h(x^*)$ is the same as that of f at x^*.

(b) Formulate and prove the analogous statement for an n-cycle.

Problem 4. Consider the following mappings

$$f(x) = a\,x \qquad\qquad g(y) = b\,y \qquad\qquad (a \neq b).$$

(a) Determine conditions on a and b for which f and g are conjugate via $y = h(x) = x^n$, where n is a positive integer.

(b) Use problem 2 (a) to infer that no diffeomorphism exists which conjugates f and g.

Problem 5. Let $f(x) = 2x\,(1-x)$, and let a, b be distinct complex numbers. Construct a map $g(y)$ which is conjugate to f, and which has fixed points at $y = a$ and $y = b$ (with the former superstable).

P.T.O.

Problem 6. Consider the two-dimensional mapping

$$x_{t+1} = x_t + \frac{2\pi\lambda}{y_{t+1}} = x_t + \frac{2\pi\lambda}{y_t + \sin x_t};$$
$$y_{t+1} = y_t + \sin x_t.$$

where x, y, λ are real numbers, $\lambda > 0$, $y_t \neq 0$, and x_t is periodic with period 2π. (Thus the phase space is a cylinder, with the circle $y = 0$ removed.)

(a) Show that the mapping has two infinite families of fixed points.

(b) Show that, for fixed λ, the number of (marginally) stable fixed points is at most finite. Determine the range of values of λ for which all fixed points are unstable.

Problem 7. Prove that the box dimension of the set

$$\{1/n^\alpha \,:\, n \in \mathbf{N}\} \qquad \alpha > 0$$

is equal to $1/(\alpha + 1)$.

Problem 8. Let $h(A, B)$ be the Hausdorff distance between two compact subsets of $\mathbf{R}$.

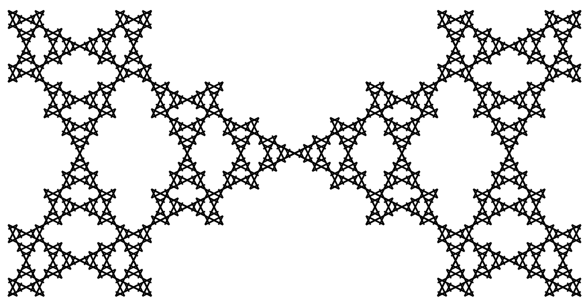
(a) Let C be the Cantor ternary set, let I be the closed unit interval, and let A be a set constituted by the point 0 and the rationals 3^{-k}, for $k = 0, 1, \ldots$, that is

$$A = \left\{0,\, 1,\, \frac{1}{3},\, \frac{1}{3^2},\, \ldots\right\}.$$

Compute $h(I, C)$, $h(I, A)$ and $h(A, C)$.

(b) Determine iterated function systems whose attractors are I, C and A, respectively.

Problem 9. Let A be the following set:



(a) After choosing coordinates, construct the iterated function system Φ whose fixed point is A.

(b) Compute the box dimension of A, hence show that A is a fractal.

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Solutions

SOLUTION TO PROBLEM 1:

Chaos is sensitive dependence on initial conditions, a mechanism that causes the orbits of a dynamical system which are initially close to each other, to separate exponentially.

Formally, a system is defined to be chaotic when its Lyapounov exponent is positive. This is the average of the logarithm of the multiplier along an orbit, which turns out to be the same for almost all orbits.

A most prominent feature of chaotic behaviour is the emergence of simple probabilistic laws, which exist alongside the extremely complex orbital motions. The latter usually include a dense set of unstable periodic orbits.

SOLUTION TO PROBLEM 2:

Notation:

$$f \overset{h}{\sim} g \qquad \Longleftrightarrow \qquad h \circ f = g \circ h$$

Reflexivity: $f \sim^{id} f$.

Simmetry: $f \sim^h g$ implies $h^{-1} \circ g = f \circ h^{-1}$, that is, $g \sim^{h^{-1}} f$. Now h^{-1} is a homeo, by definition.

Transitivity: if $f \sim^h g$ and $g \sim^k l$, then

$$f = h^{-1} \circ g \circ h = h^{-1} \circ k^{-1} \circ l \circ k \circ h = (k \circ k)^{-1} \circ l \circ (k \circ h).$$

Now, the composition of continuous bijections is a continuous bijection, so $(k \circ h)$ is a homeomorphism, and $f \sim^{k \circ h} l$.

SOLUTION TO PROBLEM 3:

(a) We have $g = h \circ f \circ h^{-1}$. Hence, letting $y = h(x)$, we have

$$g'(y) = (h^{-1})'(y)\, f'(x)\, h'(f(x))$$

If x is a fixed point, $f(x) = x$, and therefore

$$g'(y) = f'(x)[(h^{-1})'(y)\, h'(x)]$$

but the product in square brackets is unity, because

$$1 = (h^{-1} \circ h)'(x) = h'(x)\, (h^{-1})'(y).$$

(b) *Let h be a smooth conjugacy between f and g, and let $\{x_k^*\}_{k=1}^n$ be an n cycle for f. Then $\{h(x_k^*)\}_{k=1}^n$ is an n-cycle for g, with the same multiplier.*

For an n-cycle, first note that if $f \sim g$, then $f^n \sim g^n$. (with the same conjugacy function —see notes). Furthermore we know (see notes) that $\{h(x_k^*)\}_{k=1}^n$ is an n-cycle for g. Then proceed as above with f^n and g^n in place of f and g, respectively.

SOLUTION TO PROBLEM 4:

(a) We have

$$h(f(x)) = (ax)^n = g(h(x)) = bx^n.$$

So $b = a^n$, with n odd. This is a homeomorphism, but not a diffeomorphism.

(b) It suffices to note that f and g both have a unique fixed point, but with different multiplier.

SOLUTION TO PROBLEM 5:

The mapping f as an unstable fixed point at $x = 0$ and a superstable fixed point at $x = 1/2$.

Let $h(x) = mx + q$. Requiring $h(0) = b$ and $h(1/2) = a$ yields

$$h(x) = 2(a - b) + b \qquad\qquad h^{-1}(y) = \frac{y - b}{2\,(a - b)}.$$

The equation $g(y) = h(f(h^{-1}(y)))$ gives

$$g(y) = \frac{y^2 - 2ay + ab}{b - a}$$

which is the desired mapping.

SOLUTION TO PROBLEM 6:

(a) The Jacobian matrix is given by

$$J(x,y) = \begin{pmatrix} 1 - \dfrac{2\pi\lambda\cos(x)}{(y+\sin x)^2} & -\dfrac{2\pi\lambda}{(y+\sin x)^2} \\ \cos(x) & 1 \end{pmatrix}.$$

One sees that $\mathrm{Det}(J(x,y)) = 1$ (the mapping is area-preserving).

The fixed points are:

$$\frac{2\pi\lambda}{y} = 2\pi k \qquad x \equiv 0, \pi (\mathrm{mod}\ 2\pi) \qquad k \in \mathbf{Z} \quad k \neq 0.$$

They form two families:

$$(0, \lambda/k) \qquad (\pi, \lambda/k) \qquad k \neq 0.$$

(b) We compute

$$J(\pi, \lambda/k) = \begin{pmatrix} 1 + \frac{2\pi k^2}{\lambda} & +\frac{2\pi k^2}{\lambda} \\ 1 & 1 \end{pmatrix}$$

$$\mathrm{Tr}(J(\pi, \lambda/k)) = 2 + \frac{2\pi k^2}{\lambda} > 2$$

so the points $(\pi, \lambda/k)$ are all unstable.

$$J(0, \lambda/k) = \begin{pmatrix} 1 - \frac{2\pi k^2}{\lambda} & -\frac{2\pi k^2}{\lambda} \\ 1 & 1 \end{pmatrix}$$

We have

$$\mathrm{Tr}(J(0, \lambda/k)) = 2 - \frac{2\pi k^2}{\lambda} < 2. \tag{1}$$

For marginal stability, we require $\mathrm{Tr}(J(0, \lambda/k)) > -2$, that is,

$$\lambda > \frac{\pi k^2}{2}$$

which can be satisfied for at most finitely many values of k. The inequality (1) implies

$$|y| > \sqrt{\frac{\pi\lambda}{2}}.$$

which shows that the (marginally) stable fixed points are located sufficiently far away from the origin.

From (1) it also follows that for $\lambda < \pi/2$ there are no stable fixed points at all.

SOLUTION TO PROBLEM 7:

We choose boxes of size ϵ_n, where

$$\epsilon_n = \frac{1}{n^\alpha} - \frac{1}{(n+1)^\alpha} \qquad\qquad \lim_{n\to\infty} \epsilon_n = 0.$$

Using the binomial theorem, we find

$$\begin{aligned}\epsilon_n =& \frac{(n+1)^\alpha - n^\alpha}{n^\alpha (n+1)^\alpha} = \frac{n^\alpha + \alpha n^{\alpha-1} + O(n^{\alpha-2}) - n^\alpha}{n^\alpha (n+1)^\alpha} \\ =& \frac{\alpha + O(n^{-1})}{n(n+1)^\alpha} \sim \frac{\alpha}{n^{\alpha+1}}\end{aligned}$$

where, if f and g are real functions, we write $f(n) \sim g(n)$ to mean

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

To cover the point $1, 1/2^\alpha, \ldots, 1/n^\alpha$, we need n boxes of size ϵ_n. The remaining points lie in the interval $[0, 1/(n+1)^\alpha]$. To cover such an interval, we need m boxes, where

$$m = \left\lceil \frac{1}{(n+1)^\alpha \epsilon_n} \right\rceil \sim \frac{n}{\alpha}$$

and $\lceil \cdot \rceil$ is the ceiling function. Thus

$$N(\epsilon_n) = n + m \sim n(1 + 1/\alpha).$$

We compute the box dimension

$$D = \lim_{n\to\infty} \frac{\log(N(\epsilon_n))}{-\log(\epsilon_n)} = \lim_{n\to\infty} \frac{\log(1 + 1/\alpha) + \log(n)}{-\log(\alpha) + (\alpha+1)\log(n)} = \frac{1}{\alpha+1}.$$

Thus, by adjusting α, the box dimension of our set can be any real number between 0 and 1.

SOLUTION TO PROBLEM 8:

(a) One finds, in a straightforward manner

$$h(I,C) = \frac{1}{6} \qquad h(I,A) = \frac{1}{3} \qquad h(A,C) = \frac{1}{3}.$$

(b) We look for contraction mappings covering the sets I, C, and A with smaller copies of themselves. One finds

$$\begin{array}{lll} I: & w_1(x) = x/2 & w_2(x) = x/2 + 1/2 \\ C: & w_1(x) = x/3 & w_2(x) = x/3 + 2/3 \\ A: & w_1(x) = x/3 & w_2(x) = 1. \end{array}$$

Note that the IFS for the set A requires condensation.

SOLUTION TO PROBLEM 9:

(a) The set A is the union of three small copies of itself, two of which are rotated clockwise (or anti-clockwise) by $\pi/2$. It follows that the IFS for A comprises three mappings. Placing the origin in the barycentre of the A, and assuming that A has height 1, we find, using complex notation ($i = \sqrt{-1}$)

$$\begin{aligned} \Phi_1(z) &= \frac{z}{2} \\ \Phi_2(z) &= i\frac{z}{2} + \frac{3}{4} \\ \Phi_3(z) &= i\frac{z}{2} - \frac{3}{4}. \end{aligned}$$

(Alternatively, one could use affine maps of $\mathbf{R}^2$.)

(b) Because Φ consists of three maps, with the same contractivity factor $1/2$, it follows that the box dimension of A is $\log 3/\log 2$. Since 3 is not a power of 2, this number is not an integer. Hence, by definition, A is a fractal.