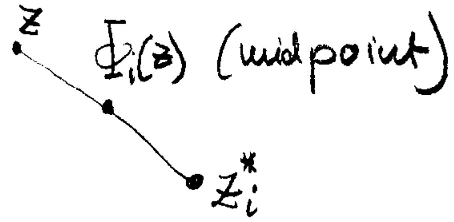
IFS

A circle is specified by 3 real numbers, a square by 8. The Julia set of a quadratic map z^2+c is specified by two, the real & imagin. part of c.
Can we define a fractal concisely? (i.e., as the <u>attractor</u> of a dynamical system?).

<u>Example</u> Let $\mathcal{H}(\mathbb{R}^2)$ be the set of all <u>compact</u>(+) subsets of $\mathbb{R}^2$, so ☺ $\in \mathcal{H}(\mathbb{R}^2)$. Let $z_i^* \in \mathbb{R}^2$ $i=1,2,3$ be 3 <u>non collinear points</u> (we use complex notation, and write $\mathcal{H}$ for $\mathcal{H}(\mathbb{R}^2)$)

$$\Phi_i(z) = \frac{z+z_i^*}{2} \quad \text{or} \quad \Phi_i(x,y) = \left(\frac{x+x^*}{2}, \frac{y+y^*}{2}\right)$$

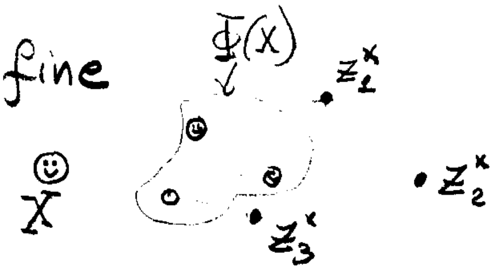
with $z^* = (x^*, y^*)$.



We have $\Phi_i(z_i^*) = z_i^*$, and z_i^* is clearly a <u>global attractor</u>.

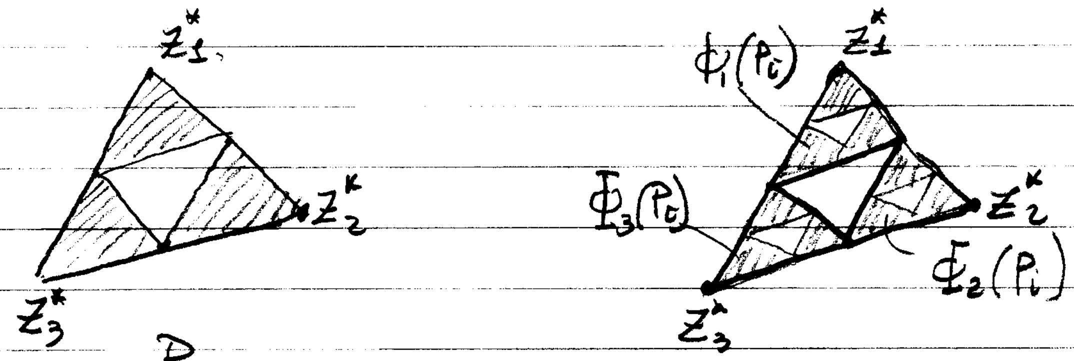
Now, for $X \in \mathcal{H}$ we define

$$\Phi(X) = \bigcup_{i=1}^{3} \Phi_i(X)$$



Let now P_t be the t-th stage in the recursive construction of the Sierpinski triangle, with vertices z_i^*

(+) Compact means <u>closed</u> and <u>bounded</u>.



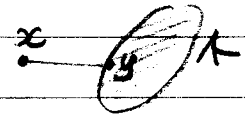
P_t $\qquad\qquad$ $P_{t+1} = \Phi(P_t)$

So, if $P_\infty = P^*$, then $P^* = \Phi(P^*)$, a fixed point!!

To express the fact that $P_t \to P^*$ we need a metric (distance on $\mathcal{H}$).

Let $x \in \mathbb{R}^2$ and $A \in \mathcal{H}(\mathbb{R}^2)$. We define the distance between x and A as

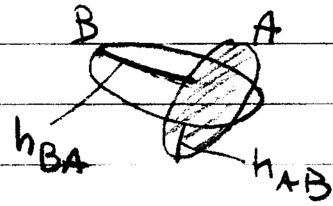
$$\rho(x, A) = \min_{y \in A} d(x, y)$$



where d is the ordinary (Euclidean) distance on $\mathbb{R}^2$

Next we define the distance between two sets $A, B \in \mathcal{H}$ as follows

$$h_{BA} = \max_{x \in B} \rho(x, A)$$



$h_{AB} \neq h_{BA}$, in general, so

$$h(A, B) = \max(h_{AB}, h_{BA})$$ the Hausdorff distance between two sets.

One can show that h is a metric on $\mathcal{H}$, i.e., for all $A, B, C \in \mathcal{H}$

$$\begin{cases} h(A,B) \geq 0 \quad \text{and } h(A,B) = 0 \text{ iff } A = B \\ h(A,B) = h(B,A) \\ h(A,C) \leq h(A,B) + h(B,C) \end{cases}$$

<u>Def</u> An <u>iterated function system</u> (IFS) on $\mathbb{R}^2$ is a mapping

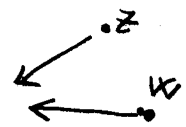
$$\Phi : \mathcal{H} \circlearrowleft \qquad \Phi(B) = \bigcup_{i=1}^{n} \Phi_i(B) \quad (*)$$

where $\Phi_i : \mathbb{R}^2 \circlearrowleft$, $i = 1, \dots, n$ is a finite collection of mappings.

The question is when Φ has a simple dynamic, i.e., a single attractive "fixed point".

<u>Def</u> A mapping $f : \mathbb{R}^2 \circlearrowleft$ is a <u>contraction mapping</u> if there exists a constant α, with $0 \leq \alpha < 1$ such that

$$d\big(f(z), f(w)\big) \leq \alpha \, d(z, w)$$



where d is the ordinary distance on $\mathbb{R}^2$.
The infimum of all such α is called the <u>contractivity factor</u> of f.

The two main results in this area are

Theorem (Dubins & Friedman, 1966). Let $\Phi = \{\phi_i\}$ an IFS on $\mathbb{R}^2$ where each Φ_i is a contraction mapping. Then Φ has a unique point attractor, whose basin of attraction is the whole of $\mathcal{H}$.

In words, there exists a compact set A^* such that

$$\Phi(A^*) = \bigcup_{i=1}^{n} \phi_i(A^*).$$

Furthermore, for each $B \in \mathcal{H}$ we have $\lim_{t\to\infty} \Phi^t(B) = A^*$

Note that A^* is made of "smaller copies of itself".

Theorem (Barnsley et al, 1984). Let Φ and A^* be as above, and let $A \in \mathcal{H}$ be such that

$$h(A, \Phi(A)) < \varepsilon$$

then

$$h(A, A^*) < \frac{\varepsilon}{1-\alpha}$$

where α is the largest of the contractivity factors of the Φ_i.

The above theorem is known as the "collage theorem". It says that if the covering of A by smaller copies of itself is not perfect but reasonably good, then A is reasonably close to A^*.

The inverse problem:

Given A^* (a "fractal picture") how do we choose an IFS whose attractor is close to A^*?

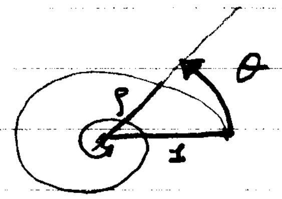
We must rely on a library of contraction mappings. The simplest cases are affine maps

$$\phi_i : z \to \underset{\substack{\uparrow \\ \text{linear}}}{M} z + \underset{\substack{\uparrow \\ \text{translation}}}{T} \qquad \text{or} \qquad \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

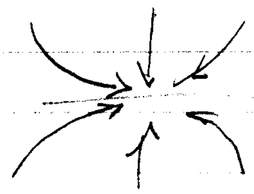
For contractivity, we require that the eigenvalues of M lie inside the unit circle. (Note: this condition only ensures that ϕ be <u>conjugate</u> to a contraction mapping, but for practical purpose this distinction is irrelevant.)

Some typical constructs:

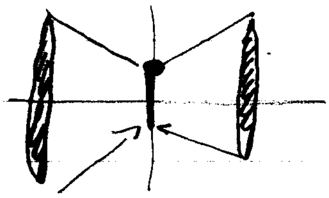
$$M = \begin{pmatrix} \rho\cos\theta & \rho\sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{pmatrix}$$



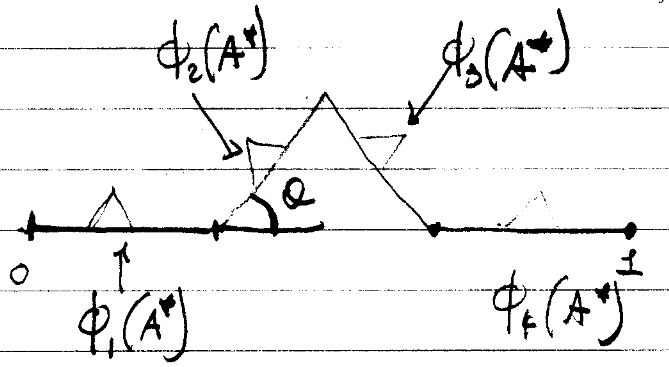
$$M = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}$$



$$M = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$$



Example The Koch curve



A^*

$\theta = \frac{\pi}{3}$

Complex notation.

$$\phi_1(z) = \frac{z}{3} \qquad \phi_4(z) = \frac{z}{3} + \frac{2}{3}$$

$$\phi_2(z) = \frac{z}{3} e^{i\frac{\pi}{3}} + \frac{1}{3} \qquad \phi_3(z) = \frac{z}{3} e^{-i\frac{\pi}{3}} + \frac{1}{3} + e^{i\frac{\pi}{3}}$$

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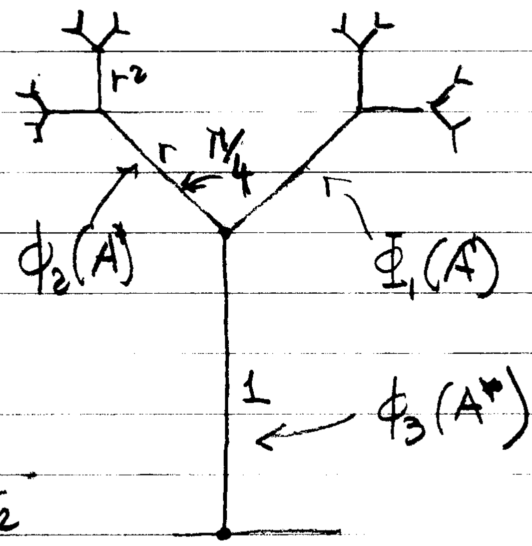
3 maps



Total height:

$$h = (1 + r^2 + r^4 + \cdots) + \frac{r}{\sqrt{2}}(1 + r^2 + r^4 + \cdots)$$

$$= \left(1 + \frac{r}{\sqrt{2}}\right) \sum_{k=0}^{\infty} (r^2)^k = \left(1 + \frac{r}{\sqrt{2}}\right) \frac{1}{1 - r^2}$$

$$\phi_1(z) = r z e^{i\frac{\pi}{4}} + i \qquad \phi_3(z) = \operatorname{Im}\left(\frac{z}{h}\right)$$

$$\phi_2(z) = r z e^{-i\pi/2} + i$$

The random algorithm

Let $\Phi = \{\phi_i\}$ $i = 1, \ldots, n$ be an IFS and let $r_1, r_2, \ldots \quad r_t \in \{1, \ldots, n\}$ be a random integer sequence. It can be shown that the dynamical system on $\mathbb{R}^2$ constructed by applying at the tth iterate the map ϕ_{r_t} has $A^* = \Phi(A^*)$ as an attractor

$$z_{t+1} = \phi_{r_t}(z_t) \qquad z_0 \text{ arbitrary.}$$

Remark if $\eta_t \in [0, 1)$ is a random variable (uniformly distributed), then

$$r_t = \lfloor n \eta_t \rfloor + 1 \qquad \lfloor \cdot \rfloor = \text{floor function}$$

is also a random variable.

Measure on fractals

We want to put shades on a fractal

Example $p^{(0)}=1$ $\qquad a=\frac{3}{4} \quad b=\frac{1}{4}$

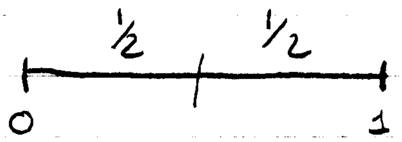
$p_1^{(1)}=\frac{1}{4}$ $\qquad p_2^{(1)}=\frac{3}{4}$ $\qquad \frac{3}{4}+\frac{1}{4}=1$

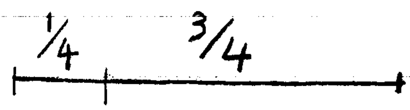
$\left(\frac{1}{4}\right)^2=\frac{1}{16}$ $\quad \frac{3}{16}$ $\quad \frac{3}{16}$ $\quad \frac{9}{16}$ $\qquad \frac{1}{16}+\frac{3}{16}+\frac{3}{6}+\frac{9}{16}=1$

$\frac{1}{64}$ $\frac{3}{64}$ $\frac{3}{64}$ $\frac{9}{64}$ $\quad \frac{3}{64}$ $\frac{9}{64}$ $\frac{9}{64}$ $\frac{27}{64}$ $\qquad \sum p_i=1$

By induction, $\sum_{i=1}^{2^k} p_i^{(k)}=1$.

The random algorithm is easily modified





Box dimension contains only info on topological structure of a fractal. If the latter is generated by a dynamical process, it will be equipped with a natural invariant measure. We use the latter to weight the boxes used to cover the given set.

Divide the phase space into boxes of equal size ε^d. Let

$$P_i = \int_{i\text{th box}} \rho(x)\,dx = \mu(i\text{th box}).$$

<u>Def</u> For every $q \in \mathbb{R}$, the <u>Renyi dimension</u> $\mathcal{D}_q$ is defined as

$$\mathcal{D}_q = \lim_{\varepsilon \to 0} \frac{1}{q-1} \frac{1}{\log \varepsilon} \log \sum_i P_i^{\,q}$$

where the sum is over all boxes for which $P_i \neq 0$.

<u>Ex</u> $q = 0$

$$\mathcal{D}_0 = \lim_{\varepsilon \to 0} \frac{1}{-\log \varepsilon} \log \sum_i 1 = \lim_{\varepsilon \to 0} \frac{1}{-\log \varepsilon} N(\varepsilon) = \text{box dim.}$$

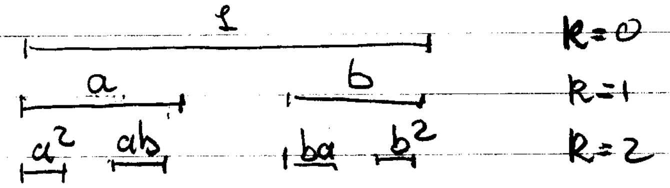
$N(\varepsilon)$: # of covering boxes ($P_i \neq 0$).

$$\mathcal{D}_2 = \lim_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \log \sum_i P_i^2$$ is the CORRELATION DIMENSION

$$\mathcal{D}_1 = \lim_{q \to 1} \mathcal{D}_q$$ is the INFORMATION DIMENSION

Example The ternary set with a probability measure.

Consider two probabilities a & b $a,b \geq 0$ $a+b=1$
Attribute product measures (probabilities) to the ternary set as follows:



as $k \to \infty$ we obtain a multifractal (fractal with a measure).
Choose boxes of size $\varepsilon_k = (1/3)^k$. The # of boxes with probability

$$P_j = a^j b^{k-j} \quad \text{is} \quad \binom{k}{j} = \frac{k!}{j!\,(k-j)!}$$

Hence

$$\sum_i P_i^q = \sum_{j=0}^{k} \binom{k}{j} a^{jq} b^{(k-j)q} = \left(a^q + b^q\right)^k$$

so that

$$D_q = \lim_{\varepsilon \to 0} \frac{1}{q-1} \frac{1}{\log \varepsilon} \log \sum_i p_i^q =$$

$$= \lim_{k \to \infty} \frac{1}{q-1} \frac{1}{\log\left(\frac{1}{3}\right)^k} \log\left(a^q + b^q\right)^k$$

$$= \lim_{k \to \infty} \frac{1}{1-q} \frac{1}{k \log 3} k \log\left(a^q + b^q\right) = \frac{\log\left(a^q + b^q\right)}{(1-q) \cdot \log 3}$$

So $D_0 = \log 2 / \log 3$. Furthermore, if $a = b = 1/2$

$$D_q = \frac{\log 2\left(\frac{1}{2}\right)^q}{(1-q)\log 3} = \frac{\log 2 - q \log 2}{(1-q)\log 3} = \frac{(1-q)\log 2}{(1-q)\log 3} = \frac{\log 2}{\log 3}$$

In this case (and only in this case) all Renyi dimensions are the same, and equal to D_0.