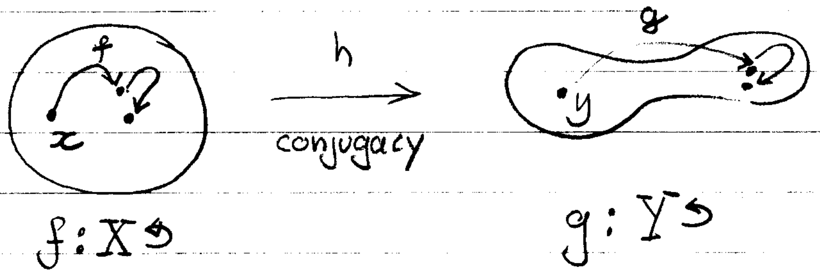
# CONJUGACY

Conjugacy = changing coordinates




<u>Def</u>: h is a <u>topological conjugacy</u> from $f: X \circlearrowleft$ to $g: Y \circlearrowleft$ if

i) h is a homeomorphism $h: X \to Y$

ii) $h \circ f = g \circ h$ (∘ denotes function composition)

Note that ii) is equivalent to $g = h \circ f \circ h^{-1}$ or $f = h^{-1} \circ g \circ h$

<u>We say that "f is topologically conjugate to g" if we can find a topological conjugacy h from f to g.</u>

Conjugacy is an <u>equivalence relation</u> (prove it!). We shall see that conjugate maps have closely related dynamics.

If h is a <u>diffeomorphism</u> we shall speak of a <u>smooth</u> conjugacy.

**Lemma 1** If $f$ and $g$ are conjugate via $h$, so are $f^n$ and $g^n$ for all $n \geq 1$.

**Proof:** $h \circ f \circ h^{-1} = g$

$$\Rightarrow g^n = \underbrace{h \circ f \circ h^{-1}}\circ \underbrace{h \circ f \circ h^{-1}} \circ \qquad \circ h \circ f \circ h^{-1}$$

$$= h \circ f \circ \underbrace{h^{-1} \circ h}_{Id} \circ f \circ \underbrace{h^{-1} \circ h}_{Id} \circ \cdots \circ f \circ h^{-1}$$

$$= h \circ f^n \circ h^{-1}$$

$\square$

**Example**

Modified Logistic map

$f: [0,1] \circlearrowleft \qquad f(x) = \mu x(1-x) \qquad \mu \in (2,4]$

Logistic map

$g: [-1,1] \circlearrowleft \qquad g(x) = 1 - \lambda x^2 \qquad \lambda \in (0,2]$

There is a conjugacy between $f$ and $g$, of the form $h(x) = c_0 + c_1 x$, with $c_0, c_1 \neq 0$. Indeed

$$h(f(x)) = c_0 + c_1 \mu x(1-x) = c_0 + c_1 \mu x - c_1 \mu x^2$$

$$g(h(x)) = 1 - \lambda(c_0 + c_1 x)^2 = 1 - \lambda c_0^2 - 2\lambda c_0 c_1 x - \lambda c_1^2 x^2.$$

Comparing coeffs of equal powers of $x$

$x^2: \quad -c_1 \mu = -\lambda c_1^2 \quad \Rightarrow \quad c_1 = \dfrac{\mu}{\lambda}$ (OK, since $\lambda \neq 0$)

$x: \quad -c_1 \mu = -2\lambda c_0 c_1 \quad \Rightarrow \quad c_0 = -\dfrac{\mu}{2\lambda}$

$x^0$: $c_0 = 1 - \lambda c_0^2$

$\lambda c_0^2 + c_0 - 1 = 0$

$$c_0 = \frac{-1 \mp \sqrt{1+4\lambda}}{2\lambda}$$

now, $\mu = -c_0 2\lambda = 1 \pm \sqrt{1+4\lambda}$

and since $\mu \in (2,4]$ we must choose the +ve sign

$$h(x) = c_0 + c_1 x = \frac{\mu}{\lambda}\left(-\frac{1}{2} + x\right) = \frac{1+\sqrt{1+4\lambda}}{\lambda}\left(x - \frac{1}{2}\right)$$

The conjugacy is singular at $\lambda = 0$ (it better be!)

**Prop.** Let $f: X\circlearrowleft$ and $g: Y\circlearrowleft$ be top. conjugated via $h$. Then

i) If $x_0, x_1, \dots$ and $y_0, y_1, \dots$ are orbits of $f$ and $g$, respectively, and if $h(x_0) = y_0$, then $h(x_t) = y_t$ for all $t \geq 0$.

ii) If $f$ has an $n$-cycle $x_0, \dots, x_{n-1}$, then $g$ has the $n$-cycle $h(x_0), \dots, h(x_{n-1})$.

**Proof**

i) $y_t = g^t(y_0) = g^t \circ h(x_0) \overset{\text{Lemma 1}}{=} h \circ f^t(x_0) = h(x_t)$

ii) $x_n = x_0 \Rightarrow h(x_n) = h(x_0) \Rightarrow y_n = y_0$.

So $y_0$ is periodic with period at most $n$.

Now $f^t(x_0) \neq x_0$ for $0 < t < n$, since the period is minimal. As $h$ is a homeo,

If $x \neq x'$, then $h(x) \neq h(x')$.
Therefore $h(f^t(x_0)) = h(x_t) \neq h(x_0)$, and so
$y_t \neq y_0 \quad 0 < t < n$

$\square$

<u>Theorem 1</u> Let $f$ and $g$ be conjugate by a <u>diffeo</u> $h$.
Then if $\rho$ is an invariant density for $f$,

$$\bar\rho(y) = \rho(h^{-1}(y)) \left|(h^{-1})'(y)\right| \qquad y = h(x)$$

is an invariant density for $g$.

<u>Pf</u> By assumption $\mathbb{P}_f \rho = \rho$. Now

$$\mathbb{P}_g \bar\rho(y) = \mathbb{P}_{h\circ f\circ h^{-1}}(y) = \sum_{\hat y \in (h\circ f\circ h^{-1})^{-1}(y)} \bar\rho(\hat y) \cdot \frac{1}{|(h\circ f\circ h^{-1})'(\hat y)|}.$$

We have $(h\circ f\circ h^{-1})^{-1} = h\circ f^{-1}\circ h^{-1}$. Letting $\hat x = h^{-1}(\hat y)$, we find

$$\hat y \in h\circ f^{-1}\circ h^{-1}(y) \iff \hat x \in f^{-1}(x)$$

$$\mathbb{P}_g \bar\rho(y) = \sum_{\hat x \in f^{-1}(x)} \rho(\hat x) \cdot \left|h^{-1'}(\cancel{h(\hat x)})\right| \cdot \frac{1}{\left| h^{-1'}(\cancel{h(\hat x)}) \cdot f'(h^{-1}(h(\hat x)) \cdot h'(f(\hat x))\right|}$$

But $f(\hat x) = x$, and so

$$\mathbb{P}_g \bar\rho(y) = \frac{1}{|h'(x)|} \mathbb{P}_f \rho(x) = \frac{\rho(x)}{|h'(x)|} = \rho(h^{-1}(y))\left|h^{-1'}(y)\right| = \bar\rho(y)$$

where we have used the fact that

$$\frac{d}{dx}\left(h^{-1}(h(x))\right) = h'(x)\cdot(h^{-1})'(y) = 1.$$

$\square$

**Theorem 2** The Lyapounov exponent is invariant under smooth conjugacy.

**Proof** Let $h \circ f = g \circ h$, with $y = f(x)$. Then

$$h'(f(x)) \cdot f'(x) = g'(y)\, h'(x)$$

$$\Lambda_f = \int_X \log|f'(x)|\, \rho(x)\, dx = \int_X \log|g'(y)|\, \rho(x)\, dx$$
$$+ \int_X \log|h'(x)|\, \rho(x)\, dx - \int_X \log|h'(f(x))|\, \rho(x)\, dx.$$

The last two integrals cancel out because $X(x) = \log|h'(x)|$ is regular ($h'(x) \neq 0$), and $\rho$ is an invariant density.(+)

As to the first integral, a change in coords. transforms $\rho$ into $\bar{\rho}$, the invariant density of $g$. Thus

$$\Lambda_f = \int_Y \log|g'(y)|\, \bar{\rho}(y)\, dy = \Lambda_g$$

$\square$

Remark: Other observables (e.g. $X(x) = x^k$) are not invariant under conjugacy.

(+) Change coordinate in the last integral, letting $y = f(x)$, and recall that $\rho$ is a fixed point of the P-F operator.

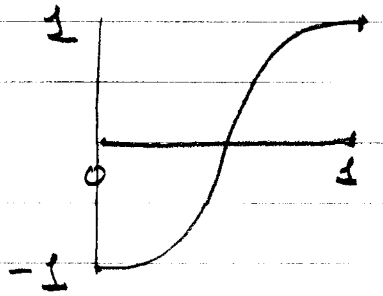
**Example** (important!)

The tent map and the Ulam map are topologically conjugate via $h(x) = -\cos(\pi x)$.

$$f(x) = \begin{cases} 2x & x \le 1/2 \\ 2(1-x) & x > 1/2 \end{cases} \quad \text{on } X = [0,1] \qquad \text{tent}$$

$$g(y) = 1 - 2y^2 \quad \text{on } Y = [-1, 1] \qquad \text{Ulam}$$

To prove it we note that $h : X \to Y$ is a homeo



Furthermore:

$$h(f(x)) = \left\{ \begin{array}{ll} -\cos \pi 2x & x \le 1/2 \\ -\cos \pi 2(1-x) & x > 1/2 \end{array} \right\} = -\cos \pi 2x \quad \forall x$$

since $-\cos \pi 2(1-x) = -\cos(-2\pi x + 2\pi) = -\cos(2\pi x)$.

$$g(h(x)) = 1 - 2\cos^2 \pi x = -\cos(2\pi x)$$

(since $\cos 2\theta = 2\cos^2\theta - 1$).

So $h \circ f = g \circ h$. [NON-SMOOTH ONLY AT BOUNDARY!]

So, for instance, the iterates $y_t$ of the Ulam map can be written as $y_t = -\cos \pi x_t$, where $x_t$ are the iterates of the Tent map

As $h$ is a diffeo on $(0,1)$, Thm 2 applies, and the Lyapounov exponents of the 2 maps coincide $\lambda_f = \lambda_g = \log 2$. (We know this already.)

The invariant density of the tent map is $\rho(x) = 1$. Thm 1 says that the invariant density of the Ulam map is

$$\bar{\rho}(y) = \rho(h^{-1}(y)) \, |(h^{-1})'(y)|$$

$$y = h(x) = -\cos \pi x \Rightarrow x = -\frac{1}{\pi} \arccos(y) = h^{-1}(y)$$

$$(h^{-1})'(y) = -\frac{1}{\pi} \frac{1}{\sqrt{1-y^2}} \quad \Rightarrow \quad \bar{\rho}(y) = 1 \cdot \frac{1}{\pi \sqrt{1-y^2}}$$

(We know this already.)

## Remarks

- Logistic map is conj. to the tent map at $\lambda = 2$ only.
- Conjugacy is often represented by the following commutative diagrams.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & f(X) \arrow[d, "h"] \\ Y \arrow[r, "g"'] & g(Y) \end{tikzcd}$$