MEASURE

The concept of the measure $\mu(E)$ of a set E is the natural generalization of concepts such as length of a segment or area of a plane figure. We develop measure on the plane.

A rectangle P on the plane is the cartesian product of two segments (open, closed, half open)

$$a \lesseqgtr x \lesseqgtr b \qquad c \lesseqgtr y \lesseqgtr d$$


P_k

Its measure is $m(P) = (b-a)(d-c)$.
An elementary set E is a set that can be represented as a finite number of disjoint rectangles P_k.

We define the measure of an elementary set to be

$$m(E) = \sum_{k=1}^{n} m(P_k).$$

It should be clear that the union, intersection and difference of elementary sets is an elementary set.

Let A now be a bounded set on the plane. We shall assume $A \subset I$, the closed unit square.
The outer measure $\mu^*(A)$ of A is the number

$$\mu^*(A) = \inf_{A \subset \bigcup_i E_i} \sum_i m(E_i)$$

where the infimum is taken over all coverings of A by finite or countable unions of elementary sets.

The <u>inner measure</u> $\mu_*(A)$ is given by

$$\mu_*(A) = 1 - \mu^*(I \setminus A)$$

If inner and outer measure coincide, the set A is said to be <u>measurable</u> and $\mu(A) = \mu^*(A) = \mu_*(A)$ is its <u>(Lebesgue) measure</u>.

<u>Remark</u> By replacing rectangles by cartesian products of n segments, one defines the Lebesgue measure on $\mathbb{R}^n$. For ordinary geometrical objects, it coincides with, length, area, volume, etc.

<u>Lemma</u>. If A is a <u>countable</u> set of points in $\mathbb{R}^n$, then $\mu(A) = 0$.

Let $A = \{x_1, x_2, x_3, \dots\}$. Since x_i is a point we cover it with a single "rectangle" P_i of measure $\delta^i = (\varepsilon^n)^i$.

Then

$$\mu(A) \le \sum_{i=1}^{\infty} \delta^i = \frac{\delta}{1-\delta}$$

Letting ε (whence δ) go to zero we obtain $\mu(A) = 0$. □

<u>Thm</u> If A_i are disjoint and measurable

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i). \quad \text{(NO PROOF)}$$

Ex Let A be the set of rational numbers in the unit interval. A is countable. Indeed we can order the elements of A by increasing denominator, and for each denominator, by increasing numerator

i	1	2	3	4	5	6	7	8	9	10	11
x_i	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{5}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{4}{5}$

So these rationals have zero measure.

Ex Let $f: \Sigma \circlearrowleft$ have finitely many (possibly zero) n-cycles, for each n. Then the set of all periodic points of f has zero measure. This is the case, e.g., when f is a polynomial.

A property of a set X is said to hold <u>almost every where</u> if the set $A \subset X$ where it does not hold has zero measure

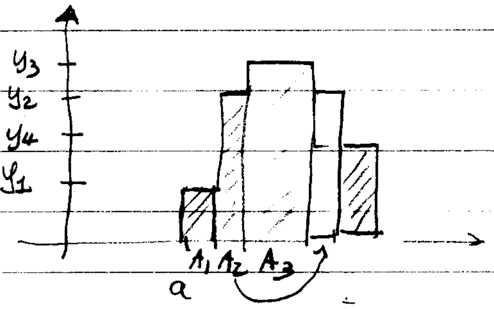
Ex <u>Almost all</u> numbers are irrational.

<u>Almost all</u> orbits of the doubling map are non-periodic.

<u>Almost all</u> triangles in the plane have unequal angles.

The Lebesgue integral

This is a generalization of Riemann integral. Let $f:[a,b]\to\mathbb{R}$ be a step function that assumes no more than countably many values $y_1, y_2, y_3, \dots$



Then $\int_a^b f(x)\,dx$ may be rewritten as $\sum_i y_i\,\mu(A_i)$ where $A_i = f^{-1}(y_i)$ is the union of disjoint intervals.

So we define
A function $f: A\to\mathbb{R}$ is <u>simple</u> if it assumes no more than countably many values $y_1, y_2, \dots$, and if the sets $A_i = f^{-1}(y_i)$ are measurable.

The Lebesgue integral of a simple function f over a set A is given by

$$\int_A f(x)\,dx = \sum_i y_i\,\mu(A_i)$$

<u>Ex</u> The function $f:[0,1]\to\mathbb{R}$ $\quad f(x) = \begin{cases} y_1 & x\in\mathbb{Q} \\ y_2 & x\notin\mathbb{Q}\end{cases}$
is simple, because it assumes two values with $\mu(A_1) = 0$ (A_1 is a countable set of pts) and $\mu(A_2) = \mu([0,1]) - \mu(A_1) = 1 - 0 = 1$.

$$\int_0^1 f(x)\,dx = y_1\cdot 0 + y_2\cdot 1 = y_2$$

The Riemann integral of f does not exist, if $y_1 \neq y_2$.

Let $f_n \to f$ uniformly on A, where f_n are simple functions. The limit

$$\lim_{n\to\infty} \int_A f_n(x)\,dx := \int_A f(x)\,dx$$

is called the <u>Lebesgue integral</u> of f on A.

If no sequence $f_n \to f$ exists, then we say that f is <u>not integrable</u>.

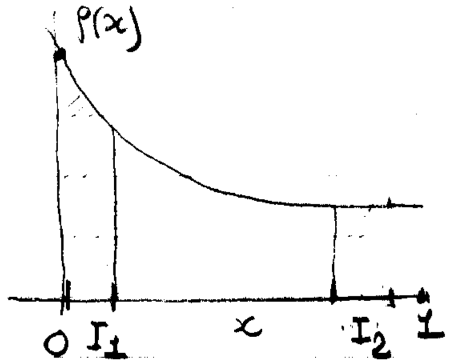
An important class of functions defined on sets, which we shall call <u>measures</u>, can be constructed from the Lebesgue measure via Lebesgue integration of non-negative functions.

Let X be a bounded set in $\mathbb{R}^n$ and let $\rho : X \to \mathbb{R}^+$ be a non-negative integrable function. For any measurable set $A \subset X$ we define

$$\mu_\rho(A) = \int_A \rho(x)\, dx \qquad (*)$$

μ_ρ is called the <u>measure</u> with <u>density</u> ρ.

<u>Ex</u>



a measure on the unit interval
$\mu_\rho(I_1) > \mu_\rho(I_2)$

<u>Ex</u> The Lebesgue measure has density $\rho(x) = 1$
Indeed such a function is simple, so that

$$\int_A 1 \cdot dx = 1 \cdot \mu(A) = \mu(A)$$

The function μ_ρ defined in $(*)$ shares with the Lebesgue measure the following properties

i) $\mu_\rho(A) \geq 0$ $\quad A \subset X \quad$ A measurable.

ii) $\mu_\rho(\bigcup_i A_i) = \sum_i \mu_\rho(A_i)$ if $A_i \cap A_j = \emptyset$ $i \neq j$.

Furthermore the function μ_g can always be normalized in such a way that

iii) $\mu_g(X) = 1$

A function satisfying i) ii) iii) is called a <u>probability measure</u> on X

Ex Let $X = [a, b]$. The density $g(x) = \frac{1}{b-a}$ define a probability measure on X, which is a normalized Lebesgue measure.